Tracts of land are often made up of irregular shapes created by geographical features such as lakes and rivers. Surveyors often measure these irregularly shaped tracts of land by dividing them into triangles. To calculate the area, side lengths, and angle measures of each triangle, they must extend their knowledge of right triangle trigonometry to include triangles that have no right angle. In this chapter you’ll learn how to do these calculations.

CHAPTER OBJECTIVES

- Given two sides and the included angle of a triangle, find by direct measurement the length of the third side of the triangle.
- Given two sides and the included angle of a triangle, derive and use the law of cosines to find the length of the third side.
- Given three sides of a triangle, find an angle measure.
- Given the measures of two sides and the included angle, find the area of the triangle.
- Given the measure of an angle, the length of the side opposite this angle, and one other piece of information about a triangle, find the other side lengths and angle measures.
- Given two sides of a triangle and a non-included angle, calculate the possible lengths of the third side.
- Given two vectors, add them to find the resultant vector.
- Given a real-world problem, identify a triangle and use the appropriate technique to calculate unknown side lengths and angle measures.
Chapter 9  Triangle Trigonometry

Overview
This chapter begins with the law of cosines, first discovered by measuring accurately drawn graphs and then by proving it with algebraic methods. Students then learn the area formula “half of side times side times sine of included angle,” which leads to the law of sines. This area formula also lays the foundation for the cross product of vectors in Chapter 12. The ambiguous case is approached through a single calculation using the law of cosines. Section 9-6, on vector addition, can be used to introduce students to the unit vectors in the x- and y-directions, although some instructors prefer to postpone vectors until Chapter 12. The chapter concludes with real-world, triangle problems where students must decide which triangle techniques to use. A cumulative review of Chapters 5 through 9 appears in Section 9-9.

Using This Chapter
This is the final chapter in Unit 2: Trigonometric and Periodic Functions. This chapter is an extension of the trigonometry students learned in geometry and earlier in this course. Consider spending some extra time on the cumulative review to make sure students grasp trigonometric concepts which form building blocks for calculus. Following this chapter, continue to Chapter 10, or, for those who studied trigonometry early in the year, return to Chapter 2.

Teaching Resources
Explorations
Exploration 9-1a:  Introduction to Oblique Triangles
Exploration 9-2:  Derivation of the Law of Cosines
Exploration 9-2a:  Angles by Law of Cosines
Exploration 9-3:  Area of a Triangle and Hero’s Formula
Exploration 9-3a:  Derivation of Hero’s Formula
Exploration 9-4:  The Law of Sines
Exploration 9-4a:  The Law of Sines for Angles

Exploration 9-5a:  The Ambiguous Case, SSA
Exploration 9-5b:  Golf Ball Problem
Exploration 9-6:  Sum of Two Displacement Vectors
Exploration 9-6a:  Navigation Vectors
Exploration 9-7a:  The Ship’s Path Problem
Exploration 9-7b:  Area of a Regular Polygon

Blackline Masters
Sections 9-8 and 9-9

Supplementary Problems
Sections 9-2, 9-3, and 9-5 to 9-8

Assessment Resources
Test 24, Sections 9-1 to 9-4, Forms A and B
Test 25, Chapter 9, Forms A and B
Test 26, Cumulative Test, Chapters 5–9, Forms A and B

Technology Resources
Dynamic Precalculus Explorations
Law of Cosines
Variable Triangle
Law of Sines

Sketchpad Presentation Sketches
Law of Cosines Present.gsp
Law of Sines Present.gsp

Activities
Sketchpad: Triangles and Squares: The Law of Cosines
Sketchpad: The Law of Sines
CAS Activity 9-2a: The Law of Sines vs. the Law of Cosines
CAS Activity 9-5a: An Alternative to the Laws of Sines and Cosines

Calculator Programs
AREGPOLY
### Standard Schedule Pacing Guide

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<th>Section</th>
<th>Suggested Assignment</th>
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<td>1–6</td>
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<tr>
<td>2</td>
<td>9-2 Oblique Triangles: The Law of Cosines</td>
<td>RA, Q1–Q10, 1, 3, 6, 7–13 odd, 14, 15, 17, 19</td>
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<tr>
<td>3</td>
<td>9-3 Area of a Triangle</td>
<td>RA, Q1–Q10, 1, 3, 7–9, 11, 13, 14</td>
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<td>4</td>
<td>9-4 Oblique Triangles: The Law of Sines</td>
<td>RA, Q1–Q10, 1–9 odd, 10, 11, 13, 14</td>
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<td>5</td>
<td>9-5 The Ambiguous Case</td>
<td>Quiz/test students on the material in Sections 9-5, assign a selection of problems not previously assigned, or use the day to recap the chapter concepts</td>
</tr>
<tr>
<td>6</td>
<td>9-6 Vector Addition</td>
<td>RA, Q1–Q10, 1, 3, 5, 7, 9</td>
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<td>9-8 Chapter Review and Test</td>
<td>R0–R7, T1–T21</td>
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<td>9</td>
<td>9-9 Cumulative Review, Chapters 5–9</td>
<td>Cumulative Review 1–18</td>
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### Block Schedule Pacing Guide

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<th>Section</th>
<th>Suggested Assignment</th>
</tr>
</thead>
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<td>1</td>
<td>9-2 Oblique Triangles: The Law of Cosines</td>
<td>RA, Q1–Q10, 1, 3, 6, 7–13 odd</td>
</tr>
<tr>
<td>2</td>
<td>9-3 Area of a Triangle</td>
<td>RA, Q1–Q10, 1, 3, 7, 8</td>
</tr>
<tr>
<td>3</td>
<td>9-4 Area of a Triangle</td>
<td>9, 11, 14</td>
</tr>
<tr>
<td>4</td>
<td>9-5 The Ambiguous Case</td>
<td>RA, Q1–Q10, 1–9 odd, 10, 14</td>
</tr>
<tr>
<td>5</td>
<td>9-6 Vector Addition</td>
<td>RA, Q1–Q10, 1, 3, 5–7, 9, 13–21 odd, 22</td>
</tr>
<tr>
<td>6</td>
<td>9-7 Real-World Triangle Problems</td>
<td>RA, Q1–Q10, 1, 3, 6, 7, 9, 13, 15, 18</td>
</tr>
<tr>
<td>7</td>
<td>9-8 Chapter Review</td>
<td>R0–R7, T1–T21</td>
</tr>
<tr>
<td>8</td>
<td>9-9 Cumulative Review, Chapters 5–9</td>
<td>Cumulative Review 1–28</td>
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<tr>
<td>10</td>
<td>9-9 Cumulative Review, Chapters 5–9</td>
<td>Cumulative Review 29–46</td>
</tr>
<tr>
<td>11</td>
<td>9-9 Cumulative Review, Chapters 5–9</td>
<td>Problem Set 10-1</td>
</tr>
<tr>
<td>12</td>
<td>Quadratic Relations and Conic Sections</td>
<td></td>
</tr>
</tbody>
</table>
Section 9-1

PLANNING

Class Time
1/2 day

Homework Assignment
Problems 1–6

Teaching Resources
Exploration 9-1a: Introduction to Oblique Triangles

Technology Resources
Exploration 9-1a: Introduction to Oblique Triangles

TEACHING

Important Terms and Concepts
Oblique triangle

Section Notes
Section 9-1 sets the stage for the development of the law of cosines in Section 9-2. You can assign this section for homework the night of the Chapter 8 test or as a group activity to be completed in class. No classroom discussion is needed before students begin the activity. You may want to adapt the problem set so that it can be done with The Geometer's Sketchpad or Fathom.

Exploration Notes
Exploration 9-1a may be assigned in place of Exploratory Problem Set 9-1. It covers much of the same material but includes a grid for students to plot side $a$ as a function of angle $A$ by hand. Encourage students to plot the points with great care. The exploration could also be used as a review sheet. Allow students 20 minutes to complete this activity.

Mathematical Overview

The Pythagorean theorem describes how to find the length of the hypotenuse of a right triangle if you know the lengths of the two legs. If the angle formed by the two given sides is not a right angle, you can use the law of cosines (an extension of the Pythagorean theorem) to find side lengths or angle measures. If two angles and a side opposite one of the angles or two sides and an angle opposite one of the sides are given, you can use the law of sines. The area can also be calculated from side and angle measures. These techniques give you a way to analyze vectors, which are quantities (such as velocity) that have both direction and magnitude. You will learn about these techniques in four ways.

GRAPHICALLY

Make a scale drawing of the triangle using the given information, and measure the length of the third side.

ALGEBRAICALLY

Law of cosines: $b^2 = a^2 + c^2 - 2ac \cos B$

Area of a triangle: $A = \frac{1}{2}ac \sin B$

NUMERICALLY

The length of side $b$: $b = \sqrt{149^2 + 237^2 - 2(149)(237) \cos 123^\circ}$

$= 341.8123... \text{ ft}$

The area of the triangle: $A = \frac{1}{2}(237)(149) \sin 123^\circ = 14,807.9868...$

$= 14,808 \text{ ft}^2$

VERBALLY

Given two sides of a triangle and the included angle, the law of cosines can be used to find the length of the third side and the sine can be used to find the area. If all three sides are given, the law of cosines can be used in reverse to find any angle measure.

Technology Notes

Exploration 9-1a in the Instructor's Resource Book has students measure the lengths of legs of various triangles and then conjecture the law of cosines. The data they gather can easily be recorded and plotted in Fathom.
### 9-1 Introduction to Oblique Triangles

You already know how to find unknown side lengths and angle measures in right triangles by using trigonometric functions. In this section you'll be introduced to a way of calculating the same kind of information if none of the angles of the triangle is a right angle. Such triangles are called **oblique triangles**.

### Objective

Given two sides and the included angle of a triangle, find by direct measurement the length of the third side of the triangle.

### Exploratory Problem Set 9-1

1. Figure 9-1a shows five triangles. Each has sides of length 3 cm and 4 cm. They differ in the measure of the angle included between the two sides. Measure the sides and angles. Do you agree with the given measurements in each case?

2. Measure \(a\), the third side of each triangle. Find the length of the third side if \(A\) were 180° and if \(A\) were 0°. Record your results in table form.

3. Store the data from Problem 2 in lists on your grapher. Make a connected plot of the data on your grapher.

4. The plot looks like a half-cycle of a sinusoid. Find the equation of the sinusoid that has the same low and high points and plot it on the same screen. Do the data really seem to follow a sinusoidal pattern?

5. By the Pythagorean theorem, \(a^2 = b^2 + c^2 \) if \(A\) is 90°. If \(A\) is less than 90°, side \(a\) is less than 5, so it seems you must subtract something from \(b^2 + c^2\) to get the value of \(a^2\). See if you can find what is subtracted!

6. What did you learn from doing this problem set that you did not know before?

![Figure 9-1a](image)

### Table

<table>
<thead>
<tr>
<th>Angle (A)</th>
<th>Side (a)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0°</td>
<td>1.0 cm</td>
</tr>
<tr>
<td>30°</td>
<td>2.1 cm</td>
</tr>
<tr>
<td>60°</td>
<td>3.6 cm</td>
</tr>
<tr>
<td>90°</td>
<td>5.0 cm</td>
</tr>
<tr>
<td>120°</td>
<td>6.1 cm</td>
</tr>
<tr>
<td>150°</td>
<td>6.8 cm</td>
</tr>
<tr>
<td>180°</td>
<td>7.0 cm</td>
</tr>
</tbody>
</table>

**Problem Notes**

*Problem 2* asks students to find \(a\) if \(A\) is 0°. Students may have trouble understanding why the result is 1 cm rather than 0 cm, because the triangle essentially “collapses” and appears to have no third side. You might suggest that students think of \(a\) as the segment connecting the endpoints of the other two segments. When \(A\) is 0°, the 3-cm and 4-cm segments lie on top of one another and \(a\) connects their right endpoints.

![Diagram](image)

*Problem 5* encourages students to try to discover the law of cosines on their own.

5. The formula is

\[ a^2 = b^2 + c^2 - 2bc \cos A, \]

that is,

\[ a^2 = b^2 + c^2 - 2bc \cos A. \]

6. Answers will vary.
Section 9-2

9-2 Oblique Triangles: The Law of Cosines

In Section 9-1, you measured the third side of triangles for which two sides and the included angle were known. Think of the three triangles with included angles 60°, 90°, and 120°.

For the right triangle in the middle, you can find the third side, \( a \), using the Pythagorean theorem.

\[ a^2 = b^2 + c^2 \]

For the 60° triangle on the left, the value of \( a^2 \) is less than \( b^2 + c^2 \). For the 120° triangle on the right, \( a^2 \) is greater than \( b^2 + c^2 \).

The equation you’ll use to find the exact length of the third side from the measures of two sides and the included angle is called the law of cosines (because it involves the cosine of the angle). In this section you’ll see why the law of cosines is true and how to use it.

**Objective**
- Given two sides and the included angle of a triangle, derive and use the law of cosines to find the length of the third side.
- Given three sides of a triangle, find an angle measure.

In this exploration you will demonstrate by measurement that the law of cosines gives the correct value for the third side of a triangle if two sides and the included angle are given.

**EXPLORATION 9-2: Derivation of the Law of Cosines**

The figure shows \( \triangle ABC \). Angle \( A \) has been placed in standard position in a \( uv \)-coordinate system.

1. The sides that include angle \( A \) have lengths \( b \) and \( c \). Write the coordinates of points \( B \) and \( C \) using \( b \), \( c \), and functions of angle \( A \).

\[ B: (u, v) = (\_\,\_\,\_\,) \]
\[ C: (u, v) = (\_\,\_\,\_\,) \]

2. Use the distance formula to write the square of the length of the third side, \( a^2 \), in terms of \( b \), \( c \), and functions of angle \( A \).

3. Simplify the equation in Problem 2 by expanding the square. Use the Pythagorean property for cosine and sine to simplify the terms containing \( \cos^2 A \) and \( \sin^2 A \).

4. \( \sqrt{4.8^2 + 2.73^2 - 2 \cdot 4.8 \cdot 2.7 \cos 115^\circ} = 6.4252 \ldots \text{ cm} \)

5. Measurements are correct.

6. “Given two sides and the included angle, the square of the side opposite the given angle equals the sum of the squares of the two given sides minus twice the product of the two given sides and the cosine of the included angle.”

Briefly: \((\text{side})^2 + (\text{side})^2 - 2(\text{side})(\text{side}) \cdot \cos(\text{of the included angle})\)

7. Answers will vary.
EXPLORATION, continued

4. The equation in Problem 3 is called the law of cosines. Show that you understand what the law of cosines says by using it to calculate the length of the third side of this triangle.

5. Measure the given sides and angle of the triangle in Problem 4. Do you agree with the given measurements? Measure the third side. Does it agree with your calculated value?

6. Describe how the unknown side in the law of cosines is related to the given sides and their included angle. Start by writing, "Given two sides and the included angle . . . ."

7. What have you learned as a result of doing this exploration that you did not know before?

Derivation of the Law of Cosines

In Exploration 9-2, you demonstrated that the square of a side \(a\) of a triangle can be found by subtracting a quantity from the Pythagorean expression \(b^2 + c^2\). Here is why this property is true. Suppose that the lengths of two sides, \(b\) and \(c\), of \(\triangle ABC\) are known, as is the measure of the included angle, \(A\) (Figure 9-2a, left).

If you construct a \(uv\)-coordinate system with angle \(A\) in standard position, as on the right in Figure 9-2a, then vertices \(B\) and \(C\) have coordinates \(B(c, 0)\) and \(C(u, v)\). By the distance formula,

\[
a^2 = (u - c)^2 + (v - 0)^2
\]

By the definitions of cosine and sine,

\[
\frac{u}{b} = \cos A \Rightarrow u = b \cos A
\]

\[
\frac{v}{b} = \sin A \Rightarrow v = b \sin A
\]

Section Notes

In this section, students derive and apply the law of cosines. One goal of this section is to have students see that the law of cosines, which works for all types of triangles, is an extension of the Pythagorean theorem, which works only for right triangles.

Note that the law of cosines is presented before the law of sines because it is a more reliable technique. The sign of \(\cos A\) indicates whether \(A\) is obtuse or acute. No such information can be obtained by using the law of sines.

The problems in this section involve using the law of cosines in cases where two sides and an included angle are known (SAS) or where three sides are known (SSS). It can also be used in the ambiguous case, in which two sides and a non-included angle are known (SSA). This case will be discussed in Section 9-5.

Example 1 applies the law of cosines to find a missing side when two sides and an included angle are known. In Example 2, the law of cosines is used to find a missing angle when three sides are known.

In \(\triangle XYZ\) you may want to encourage students to write the \(Z\) with a bar through it so that the \(Z\) is not confused with a 2.

The lengths given in Example 3 do not form a triangle, so the law of cosines yields no solution. Some students may realize immediately that a triangle cannot be formed because the lengths do not satisfy the triangle inequality.

This section provides an excellent opportunity to review some of the triangle concepts students learned in geometry. Here are some topics you might discuss.

- The fact that the largest angle in a triangle is opposite the longest side and that the smallest angle is opposite the shortest side
- The triangle inequality
- Tests for determining whether an angle is right, acute, or obtuse (see Problem 18)
- Conventions for naming sides and angles (in some texts, angles are denoted by the Greek letters alpha, \(\alpha\); beta, \(\beta\); and gamma, \(\gamma\))
**Differentiating Instruction**
- Pass out the list of Chapter 9 vocabulary, available at www.keypress.com/keyonline, for ELL students to look up and translate in their bilingual dictionaries.
- Have ELL students find out what the law of cosines is called in their own language. This will help students understanding the meaning of law in the context of mathematics.
- Have students enter the law of cosines in their journals three times, once with \( a \) on the left side, once with \( b \), and once with \( c \). Figuring out the other two forms will help students learn the formula.
- The Reading Analysis should be done in pairs.
- ELL students will probably need language support on Problems 14–18.

**Additional Exploration Notes**
*Exploration 9-2a* has students first find angles by using the law of cosines and then verify their results by measuring. This activity can be done by groups of students in class or assigned for homework. Allow about 15 minutes for this exploration.

**Technology Notes**
- **Problem 16:** Geometric Derivation of the Law of Cosines Problem asks students to experiment with the Dynamic Precalculus Exploration at www.keymath.com/precalc in order to derive the law of cosines.

**Applications of the Law of Cosines**
You can use the law of cosines to calculate the measure of either a side or an angle. In each case, different parts of a triangle are given. Watch for what these “givens” are.

**Example 1**
In \( \triangle PMF \), \( M = 127^\circ \), \( p = 15.78 \) ft, and \( f = 8.54 \) ft. Find the measure of the third side, \( m \).

**Solution**
First, sketch the triangle and label the sides and angles, as shown in Figure 9-2b. (It does not need to be accurate, but it must have the right relationship among sides and angles.)

**Activity:** Triangles and Squares: The Law of Cosines in the Instructor’s Resource Book provides another visual proof of the law of cosines. Students build a square with side length equal to the length of the longest leg of an obtuse triangle, and then they compare areas of different figures that result.

**CAS Activity 9-2a:** The Law of Sines vs. the Law of Cosines in the Instructor’s Resource Book has students explore the various challenges of using the laws of sines and cosines. As students using a CAS will discover, the law of cosines is more consistent when the algebraic calculations can be done using a grapher. Allow 20–25 minutes.

**PROPERTY: The Law of Cosines**
In triangle \( \triangle ABC \) with sides \( a \), \( b \), and \( c \),

\[
A^2 = b^2 + c^2 - 2bc \cos A
\]

**Notes:**
- If the angle measures 90°, the law of cosines reduces to the Pythagorean theorem, because \( \cos 90^\circ \) is zero.
- If angle \( A \) is obtuse, \( \cos A \) is negative. So you are subtracting a negative number from \( b^2 + c^2 \), giving the larger value for \( a^2 \), as you found in Section 9-1.
- You should not jump to the conclusion that the law of cosines gives an easy way to prove the Pythagorean theorem. Doing so would involve circular reasoning, because the Pythagorean theorem (in the form of the distance formula) was used to derive the law of cosines.
- A capital letter is used for the vertex, the angle at that vertex, or the measure of that angle, whichever is appropriate. If confusion results, you can use the symbols from geometry, such as \( m\angle A \) for the measure of angle \( A \).
EXAMPLE 2
In \( \triangle XYZ \), \( x = 3 \text{ m} \), \( y = 7 \text{ m} \), and \( z = 9 \text{ m} \). Find the measure of the largest angle.

**Solution**

Make a sketch of the triangle and label the sides, as shown in Figure 9-2c.

Recall from geometry that the largest side is opposite the largest angle, in this case, \( Z \). Use the law of cosines with this angle and the two sides that include it.

\[
9^2 = 7^2 + 3^2 - 2 \cdot 7 \cdot 3 \cos Z \\
81 = 49 + 9 - 42 \cos Z \\
81 - 49 - 9 = -42 \cos Z \\
-0.5476 = \cos Z \\
Z = \arccos(-0.5476...) = \cos^{-1}(-0.5476...) = 123.2038... = 123.2^\circ
\]

Note that \( \arccos(-0.5476...) = \cos^{-1}(-0.5476...) \) in Example 2 because there is only one value of an arccosine between 0° and 180°, the range of angles possible in a triangle.

EXAMPLE 3
Suppose that the lengths of the sides in Example 2 had been \( x = 3 \text{ m} \), \( y = 7 \text{ m} \), and \( z = 11 \text{ m} \). What would the measure of angle \( Z \) be in this case?

**Solution**

Write the law of cosines for side \( z \), the side that is opposite angle \( Z \).

\[
11^2 = 7^2 + 3^2 - 2 \cdot 7 \cdot 3 \cos Z \\
121 = 49 + 9 - 42 \cos Z \\
121 - 49 - 9 = -42 \cos Z \\
-1.5 = \cos Z
\]

There is no such triangle.

The geometric reason why there is no solution in Example 3 is that no two sides of a triangle can sum to less than the third side. Figure 9-2d illustrates this fact. The law of cosines signals this inconsistency algebraically by giving a cosine value outside the interval \([-1, 1]\).

---

### CAS Suggestions

For triangles that can be solved using the law of cosines, a **Solve** command is ideal for differentiating between triangles that are impossible, possible, and ambiguous. When using a CAS, students can insert variables for any unknown quantity and solve the resulting equation. The figure shows Examples 1, 2, and 3.

Note that the CAS gives both algebraic answers to Example 1 even though only the positive solution makes sense in context. Students should learn to make this distinction. It is helpful to restrict the angle values in these examples to those suitable for triangles. This can be done using a \( | \) command as shown in lines 2 and 3 of the previous figure.

An alternative use of the law of cosines is to define a **Solve** command using the generic law of cosines formula and substituting for all values using a \( | \) command. Some students may prefer to use this method because they see the law of cosines in its familiar form. (In the figure the command is executed twice so that you can see the beginning and end of the command line.) Finally, notice that when the side lengths are entered with units, the results also include units.
Supplementary problems for this section are available at [www.keypress.com](http://www.keypress.com/keyonline).

Remind students to store intermediate answers without rounding. Only final results should be rounded.

Q1. \( \frac{i}{u} \)  Q2. \( \frac{i}{q} \)
Q3. 1  Q4. \( u \sin i \)
Q5. \( \sqrt{u^2 - q^2} \)  Q6. \( \tan \frac{q}{i} \)
Q7. Sinusoidal axis
Q8. \( \cos 35^\circ \cos 42^\circ + \sin 35^\circ \sin 42^\circ \)
Q9. Horizontal dilation by a factor of \( \frac{1}{5} \)
Q10. \( 2 \sin x \cos x \)

Problems 1–12 are straightforward and provide practice with the law of cosines. In Problems 9 and 10, a triangle cannot be formed from the given side lengths. Students may discover this by using the triangle inequality or by attempting to apply the law of cosines.

1. \( r = 3.98 \text{ cm} \)
2. \( d = 5.05 \text{ in} \)
3. \( r = 4.68 \text{ ft} \)
4. \( k = 13.97 \text{ m} \)

In Problems 5–12 you can use a \( \text{command to prevent unnecessary solutions.} \)

5. \( U = 28.96^\circ \)
6. \( G = 92.87^\circ \)
7. \( T = 134.62^\circ \)
8. \( E = 102.64^\circ \)
9. This is not a possible triangle, because \( 7 + 5 < 13 \).
10. This is not a possible triangle, because \( 6 + 3 < 12 \).
11. \( O = 90^\circ \)
12. \( Q = 90^\circ \)

Problems 13 and 14 are ideal to do using The Geometer’s Sketchpad. If the program is unavailable, students can use a protractor, compass, and ruler to construct the triangles and find the unknown measures. Students’ measurements should agree with the result given by the law of cosines. Centimeter graph paper from the blackline master in the Instructor’s Resource Book may be used.

13a. \( r = 4.0 \text{ cm}, p = 4.0 \text{ cm}, m = 5.0 \text{ cm}, \)
\( R = 51^\circ \)
13b. \( m = 5.0 \text{ cm}, e = 6.0 \text{ cm}, g = 8.0 \text{ cm}, \)
\( G = 93^\circ \)
14a. \( \approx 192.7 \text{ ft} \)
14b. \$722.72
14c. \$975.67
15. \( \cos^{-1} \frac{15^2 + 21^2 - 33^2}{2 \cdot 15 \cdot 21} \approx 132.2^\circ \)
14. **Fence Problem**: Mattie works for a fence company. She has the job of pricing a fence to go across a triangular lot at the corner of Alamo and Heights Streets, as shown in Figure 9-2f. The streets intersect at a 65° angle. The lot extends 200 ft from the intersection along Alamo and 150 ft from the intersection along Heights.

![Image of fence and streets](Figure 9-2f)

**a.** How long will the fence be?
**b.** How much will it cost her company to build the fence if fencing costs $3.75 per foot?
**c.** What price should she quote to the customer if the company is to make a 35% profit?

15. **Flight Path Problem**: Sam flies a helicopter to drop supplies to stranded flood victims. He will fly from the supply depot, S, to the drop point, P. Then he will return to the helicopter’s base at B, as shown in Figure 9-2g. The drop point is 15 mi from the supply depot. The base is 21 mi from the drop point. It is 33 mi between the supply depot and the base. Because the return flight to the base will be made after dark, Sam wants to know in what direction to fly. What is the angle between the two paths at the drop point?

![Image of flight path](Figure 9-2g)


17. **Derivation of the Law of Cosines Problem**: Figure 9-2h shows △XYZ with angle Z in standard position. The sides that include angle Z are 4 units and 5 units long, as shown. Find the coordinates of points X and Y in terms of 4, 5, and angle Z. Then use the distance formula, appropriate algebra, and trigonometry to show that

\[ z^2 = x^2 + y^2 - 2xy \cos Z \]

**a.** Explain how the law of cosines allows you to make a quick test to see whether angle X is acute, right, or obtuse, as shown in this box:

**PROPERTY: Test for the Size of an Angle in a Triangle**

In △XYZ:
- If \( x^2 < y^2 + z^2 \), then angle X is an acute angle.
- If \( x^2 = y^2 + z^2 \), then angle X is a right angle.
- If \( x^2 > y^2 + z^2 \), then angle X is an obtuse angle.

**b.** Without using your calculator, find whether angle X is acute, right, or obtuse if \( x = 7 \) cm, \( y = 5 \) cm, and \( z = 4 \) cm.

18. **Acute, Right, or Obtuse Problem**: The law of cosines states that in △XYZ

\[ x^2 = y^2 + z^2 - 2yz \cos X \]

**a.** Show that \( x^2 < y^2 + z^2 \) if and only if \( \cos X > 0 \), which means angle X is acute.
**b.** Show that \( x^2 = y^2 + z^2 \) if and only if \( \cos X = 0 \), which means angle X is right.
**c.** Show that \( x^2 > y^2 + z^2 \) if and only if \( \cos X < 0 \), which means angle X is obtuse.

Problem 16 gives students an opportunity to explore a geometrical derivation of the law of cosines using a Dynamic Precalculus Exploration at www.keymath.com/precalc.

**16.** Answers will vary.

19. **Problem Set 9-2**

51° 4 cm, \( \sin 51° = 0.7746 \)

\( \cos 51° = 0.6367 \)

\( \sin 18° = 0.3090 \)

\( \cos 18° = 0.9511 \)

5. **Problem 5a**: If \( x^2 < y^2 + z^2 \), then \( y^2 + z^2 - 2yz \cos X < y^2 + z^2 \), which happens exactly when \( \cos X > 0 \); so X is acute.

6. **Problem 5b**: If \( x^2 = y^2 + z^2 \), then \( y^2 + z^2 - 2yz \cos X = y^2 + z^2 \), which happens exactly when \( \cos X = 0 \); so X is right.

7. **Problem 5c**: If \( x^2 > y^2 + z^2 \), then \( y^2 + z^2 - 2yz \cos X > y^2 + z^2 \), which happens exactly when \( \cos X < 0 \); so X is obtuse.

8. **Problem 6a**: \( 7^2 = 49 > 41 = 5^2 + 4^2 \);

9. **Problem 6b**: so X is obtuse.

**Additional CAS Problems**

1. In the triangles shown below, determine the measures of the unknown sides.

   ![Triangle 1](a.png)
   ![Triangle 2](b.png)

2. The perimeter of a triangle is 20 units and the length of one side is 9.5 units. If the angle opposite the given side measures 2 radians, what are the lengths of the other two sides?
Section 9-3

Class Time
1 day

Homework Assignment
RA, Q1–Q10, Problems 1, 3, 7–9, 11, 13, 14

Teaching Resources
Exploration 9-3: Area of a Triangle and Hero’s Formula
Exploration 9-3a: Derivation of Hero’s Formula
Supplementary Problems

Technology Resources
Problem 11: Variable Triangle Problem
Exploration 9-3: Area of a Triangle and Hero’s Formula

9-3 Area of a Triangle

Recall from earlier math classes that the area of a triangle equals half the product of the base and the altitude. In this section you’ll learn how to find this area from two side lengths and the included angle measure. This is the same information you use in the law of cosines to calculate the length of the third side.

Objective
Given the measures of two sides and the included angle, or the measures of all three sides, find the area of the triangle.

In this exploration you will discover a quick method for calculating the area of a triangle from the measures of two sides and the included angle.

Exploration 9-3: Area of a Triangle and Hero’s Formula

For Problems 1–3, \( \triangle XYZ \) has sides \( y = 8 \text{ cm} \) and \( z = 7 \text{ cm} \) and included angle \( X \) with measure \( 38^\circ \).

1. Do you agree with the given measurement for \( y \)? for \( z \)? for \( \angle X \)?
2. Use the given measurements to calculate altitude \( h \). Measure \( h \). Does it agree with the calculation?
3. Recall from geometry that the area of a triangle is \( \frac{1}{2} \text{(base)(altitude)}. \) Find the area of \( \triangle XYZ \).
4. By substituting \( z \sin X \) for the altitude in Problem 3 you get
   
   \[
   \text{Area} = \frac{1}{2}yz \sin X
   \]
   or, in general,
   
   \[
   \text{Area} = \frac{1}{2} \text{(side)(side)(sine of included angle)}
   \]
   Sketch a triangle with sides 43 m and 51 m and included angle \( 143^\circ \). Use this area formula to find the area of this triangle.

5. Find the measure of angle \( A \) using the law of cosines. Store the answer without rounding.
6. Use the unrounded value of \( A \) and the area formula of Problem 4 to find the area of \( \triangle ABC \).
7. Calculate the semiperimeter (half the perimeter) of the triangle, \( s = \frac{1}{2}(a + b + c) \).
8. Evaluate the quantity \( \sqrt{s(s - a)(s - b)(s - c)} \). What do you notice about the answer?
9. Use Heron’s formula, namely,
   
   \[
   \text{Area} = \sqrt{s(s - a)(s - b)(s - c)}
   \]
   to find the area of this triangle.

10. What did you learn as a result of doing this exploration that you did not know before?

   \[
   \text{Area} = 27.9284...
   \]

   \[
   s = 13
   \]

   \[
   27.9284...; \text{ The answer is the same as the area calculated in Problem 6.}
   \]

   \[
   \text{Area} = 548.2773... \approx 548.3 \text{ ft}^2
   \]

   \[
   \text{Answers will vary.}
   \]

Section Notes

In this section, the traditional formula for the area of a triangle, \( A = \frac{1}{2}bh \), is transformed into one involving trigonometry, \( A = \frac{1}{2}bc \sin A \). The transformation is straightforward.

The derivation of the area formula given is for an acute angle \( A \). The formula also works if \( A \) is obtuse. You may want to show students the derivation.
The following is a derivation of the area formula you discovered in Exploration 9-3. Figure 9-3a shows \( \triangle ABC \) with base \( b \) and altitude \( h \).

\[
\text{Area} = \frac{1}{2}bh
\]

From geometry, area equals half base times altitude.

\[
\text{Area} = \frac{1}{2}b(c \sin A)
\]

Because \( \sin A = \frac{h}{c} \).

\[
\text{Area} = \frac{1}{2}bc \sin A
\]

\[\textbf{PROPERTY: Area of a Triangle}\]

In \( \triangle ABC \),

\[\text{Area} = \frac{1}{2}bc \sin A\]

\[\text{Verbally: The area of a triangle equals half the product of two of its sides and the sine of the included angle.}\]

**EXAMPLE 1**

**SOLUTION**

In \( \triangle ABC \), \( a = 13 \text{ in.} \), \( b = 15 \text{ in.} \), and \( C = 71^\circ \). Find the area of the triangle.

\[
\text{Sketch the triangle to be sure you're given two sides and the included angle (Figure 9-3b).}
\]

\[
\text{Area} = \frac{1}{2}(13)(15)\sin 71^\circ
\]

\[= 92.1880\ldots \approx 92.19 \text{ in.}^2\]

**EXAMPLE 2**

**SOLUTION**

Find the area of \( \triangle DH \) if \( j = 5 \text{ cm} \), \( d = 7 \text{ cm} \), and \( h = 11 \text{ cm} \).

\[
\text{Sketch the triangle to give yourself a picture of what has to be done (Figure 9-3c).}
\]

\[
h^2 = j^2 + d^2 - 2jd \cos H
\]

\[\cos H = \frac{j^2 + d^2 - h^2}{2jd}\]

\[= \frac{5^2 + 7^2 - 11^2}{2(5)(7)} = -0.6714\ldots\]

Solve for \( \cos H \).

\[H = \arccos(-0.6714\ldots) = \cos^{-1}(-0.6714\ldots) = 132.1774\ldots^\circ\]

\[\text{Area} = \frac{1}{2}(5)(7)\sin 132.1774\ldots^\circ = 12.9687\ldots \approx 12.97 \text{ cm}^2\]

\[\textbf{Hero's Formula}\]

It is possible to find the area of a triangle directly from the lengths of three sides without going through the angle calculations of Example 2. The method uses \textit{Hero's formula}, named after Hero of Alexandria, who lived around 100 B.C.E.

This derivation is an ideal opportunity to remind students that an altitude of a triangle may fall outside the triangle. Example 1 involves direct substitution into the new area formula, whereas Example 2 requires first finding an angle using the law of cosines.

In Example 3, Hero's formula, named after Hero of Alexandria, who lived about 100 B.C.E., is used to find the area of a triangle given the length of three sides.

**Differentiating Instruction**

- Have students research the name for \textit{Hero's formula} in their own language.
- Depending on the language skills of your class, students may be able to do \textit{Exploration 9-3} individually or may need to do it in pairs.
- Have students write the formula for the area of a triangle as they have previously learned it in their journals, and then have them write all three forms of the formula on page 451. Finally, have them write the area of a triangle property in their own words.
- Most students will be able to do the Reading Analysis individually, but provide assistance if needed.
- You may need to provide support with the language in \textit{Problems 10–14}.

\[\textbf{Additional Exploration Notes}\]

\textit{Exploration 9-3a} requires students to apply the law of cosines, the formula \(\text{Area} = \frac{1}{2}bc \sin A\), and Hero’s formula and then guides them through the derivation of Hero’s formula. You might use this activity as a take-home assignment or as a group quiz. Allow at least 20 minutes.
Technology Notes

Problem 11 asks students to find the area of a triangle as a function of one angle, when the two legs creating that angle are of fixed lengths. Students are asked to view a Dynamic Precalculus Exploration of this triangle at www.keymath.com/precalc. This problem is related to Exploration 9-1a in the Instructor’s Resource Book, in the sense that both activities begin with a triangle that has two fixed legs and a variable included angle \( \theta \), and they express some parameter as a function of \( \theta \).

Exploration 9-3 guides students through deriving a formula for area, given the lengths of two sides and the measure of the included angle. Sketchpad can be used to test conjectures about formulas against the area as calculated by the software.

CAS Suggestions

Area problems can be solved similarly to the law of cosines problems as described in the CAS Suggestions on page 447 in Section 9-2.

One approach to the trigonometric formula for the area of a triangle is to define functions using a CAS. Students can define the area function as shown in the figure, then use the function to evaluate the area.

If your students are using units in a problem, but get English units when they are expecting metric, check the system settings or convert the units directly using the conversion command.

Students may also find it helpful to define Hero’s formula as a function. Use the formula for the semiperimeter instead of a fourth variable to eliminate extra calculations. It is particularly nice that the CAS will return an exact value.

PROPERTY: Hero’s Formula

In \( \triangle ABC \), the area is given by

\[
\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}
\]

where \( s \) is the semiperimeter (half the perimeter), \( \frac{1}{2}(a + b + c) \).

**Example 3**

Find the area of \( \triangle IDH \) in Example 2 using Hero’s formula. Confirm that you get the same answer as in Example 2.

**Solution**

\[
s = \frac{1}{2}(5 + 7 + 11) = 11.5
\]

\[
\text{Area} = \sqrt{11.5(11.5 - 5)(11.5 - 7)(11.5 - 11)} = \sqrt{168.1875} = 12.9687... \approx 12.97 \text{ cm}^2
\]

Agrees with Example 2.

**Problem Set 9-3**

**Reading Analysis**

From what you have read in this section, what do you consider to be the main idea? In what way is the area formula \( \text{Area} = \frac{1}{2}bc \sin A \) related to the formula \( \text{Area} = \frac{1}{2}(\text{base})(\text{height}) \)? What formula allows you to calculate the area of a triangle directly, from three given side lengths?

**Quick Review**

Problems Q1–Q5 refer to Figure 9-3d.

Q1. State the law of cosines using angle \( R \).
Q2. State the law of cosines using angle \( S \).
Q3. State the law of cosines using angle \( T \).
Q4. Express \( \cos T \) in terms of sides \( r \), \( s \), and \( t \).
Q5. Why do you need only the function \( \cos^{-1} \), not the relation \( \arccos \), when using the law of cosines to find an angle?
Q6. When you multiply two sinusoids with very different periods, you get a function with a varying ______.
Q7. What is the first step in proving that a trigonometric equation is an identity?
Q8. Which trigonometric functions are even functions?
Q9. If angle \( \theta \) is in standard position, then ________ is the definition of ________.

Q10. In the composite argument properties, \( \cos(x + y) = \) ________.

For Problems 1–4, find the area of the indicated triangle.

1. \( \triangle ABC \), if \( a = 5 \text{ ft} \), \( b = 9 \text{ ft} \), and \( C = 14^\circ \)
2. \( \triangle ABC \), if \( b = 8 \text{ m} \), \( c = 4 \text{ m} \), and \( A = 67^\circ \)
3. \( \triangle RST \), if \( r = 4.8 \text{ cm} \), \( t = 3.7 \text{ cm} \), and \( S = 43^\circ \)
4. \( \triangle XYZ \), if \( x = 34.19 \text{ yd} \), \( z = 28.65 \text{ yd} \), and \( Y = 138^\circ \)

For Problems 5–7, use Hero’s formula to calculate the area of the triangle.

5. \( \triangle ABC \), if \( a = 6 \text{ cm} \), \( b = 9 \text{ cm} \), and \( c = 11 \text{ cm} \)
6. \( \triangle XYZ \), if \( x = 50 \text{ yd} \), \( y = 90 \text{ yd} \), and \( z = 100 \text{ yd} \)
7. \( \triangle DEF \), if \( d = 3.7 \text{ in} \), \( e = 2.4 \text{ in} \), and \( f = 4.1 \text{ in} \)

8. Comparison of Methods Problem: Reconsider Problems 1 and 7.

a. For \( \triangle ABC \) in Problem 1, calculate the length of the third side using the law of cosines. Store the answer without rounding. Then find the area using Hero’s formula. Do you get the same answer as in Problem 1?

b. For \( \triangle DEF \) in Problem 7, calculate the measure of angle \( D \) using the law of cosines. Store the answer without rounding. Then find the area using the area formula as in Example 2. Do you get the same answer as in Problem 7?

**Solution**

\[
\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}
\]

where \( s \) is the semiperimeter (half the perimeter), \( \frac{1}{2}(a + b + c) \).
9. **Hero’s Formula and Impossible Triangles Problem:** Suppose someone tells you \( \triangle ABC \) has sides \( a = 5 \text{ cm}, b = 6 \text{ cm}, \) and \( c = 13 \text{ cm} \).

   a. Explain why there is no such triangle.
   b. Apply Hero’s formula to the given information. How does Hero’s formula allow you to detect that there is no such triangle?

10. **Lot Area Problem:** Sean works for a real estate company. The company has a contract to sell the triangular lot at the corner of Alamo and Heights Streets (Figure 9-3c). The streets intersect at a 65° angle. The lot extends 200 ft from the intersection along Alamo and 150 ft from the intersection along Heights.

   a. Find the area of the lot.
   b. Land in this neighborhood is valued at $35,000 per acre. An acre is 43,560 ft\(^2\). How much is the lot worth?
   c. The real estate company will earn a commission of 6% of the sales price. If the lot sells for what it is worth, how much will the commission be?

![Figure 9-3c](image)

11. **Variable Triangle Problem:** Figure 9-3f shows angle \( \theta \) in standard position in a \( uv\)-coordinate system. The fixed side of the angle is 3 units long, and the rotating side is 4 units long. As \( \theta \) increases, the area of the triangle shown in the figure is a function of \( \theta \).

![Figure 9-3f](image)

The names given to the \textit{trigarea} and \textit{hero} functions are entirely arbitrary. Some CAS graphers require the use of standard function names. Students can name functions in whatever way helps them remember best.

**Q1.** \( r^2 = s^2 + t^2 - 2st \cos R \)

**Q2.** \( s^2 = r^2 + t^2 - 2rt \cos S \)

**Q3.** \( t^2 = r^2 + s^2 - 2rs \cos T \)

**Q4.** \( \cos T = \frac{r^2 + s^2 - t^2}{2rs} \)

**Q5.** The other values are either negative or greater than 180° and therefore could not be angles of a triangle.

**Q6.** Amplitude

**Q7.** Start with the more complicated side and try to simplify it to equal the other side.

**Q8.** Cosine and secant

**Q9.** \( \cos \theta \)

**Q10.** \( \cos x \cos y - \sin x \sin y \)

See page 1018–1019 for answers to Problems 1–14 and CAS Problems 1 and 2.
**Section 9-4**

**Oblique Triangles: The Law of Sines**

Because the law of cosines involves all three sides of a triangle, you must know at least two of the sides to use it. In this section you'll learn the law of sines, which lets you calculate a side length of a triangle if only one side and two angles are given.

**Objective**

Given the measure of an angle, the length of the side opposite this angle, and one other piece of information about a triangle, find the other side lengths and angle measures.

In this exploration you will use the ratio of a side length to the sine of the opposite angle to find the measures of other parts of a triangle.

**EXPLORATION 9-4: The Law of Sines**

1. In \( \triangle XYZ \), are the following measurements correct?
   - \( y = 6.0 \text{ cm} \)
   - \( z = 7.0 \text{ cm} \)
   - \( Y = 57^\circ \)
   - \( Z = 78^\circ \)

2. Assuming that the measurements in Problem 1 are correct, calculate these ratios:
   - \( \frac{y}{\sin Y} \)
   - \( \frac{z}{\sin Z} \)

3. The law of sines states that within a triangle, the ratio of the length of a side to the sine of the opposite angle is constant. Do the calculations in Problem 2 seem to confirm this property?

4. Measure angle \( X \).

5. Assuming that the law of sines is correct,
   \[ \frac{x}{\sin X} = \frac{y}{\sin Y} \]

   Use this information and the measured value of \( X \) to calculate length \( x \).

6. Measure side \( x \). Does your measurement agree with the calculated value in Problem 5?

7. For \( \triangle ABC \), use the area formula to write the area three ways:
   - a. Involving angle \( A \)
   - b. Involving angle \( B \)
   - c. Involving angle \( C \)

8. The area of a triangle is independent of the way you calculate it, so all three expressions in Problem 7 are equal to each other. Write a three-part equation expressing this fact.

9. Divide all three "sides" of the equation in Problem 8 by whatever is necessary to leave only the sines of the angles in the numerators. Simplify.

10. The statements are equivalent because if the parts of an equation are equal and nonzero, then the reciprocals of the parts of the equation are equal to each other and nonzero.

11. Answers will vary.

**Section Notes**

The law of sines can be applied when two angles and a non-included side of a triangle are known (AAS) or when two angles and an included side are known (ASA). It can also be used with caution in the ambiguous case, in which two sides and a non-included angle are known (SSA). The ambiguous case is covered in Section 9-5.
EXPLORATION, continued

10. The equation you should have gotten in Problem 9 is the law of sines. Explain why it is equivalent to the law of sines as written in Problem 5.

11. What did you learn as a result of doing this exploration that you did not know before?

In Exploration 9-4, you demonstrated that the law of sines is correct and used it to find an unknown side length of a triangle from information about other sides and angles. The law of sines can be proved with the help of the area formula from Section 9-3.

Figure 9-4a shows \( \triangle ABC \). In the previous section you found that the area is equal to \( \frac{1}{2}bc \sin A \). The area is constant no matter which pair of sides and included angle you use.

\[
\frac{1}{2}bc \sin A = \frac{1}{2}ac \sin B = \frac{1}{2}ab \sin C
\]

Set the areas equal.

\[
bc \sin A = ac \sin B = ab \sin C
\]

Multiply by 2.

\[
\frac{bc \sin A}{abc} = \frac{ac \sin B}{abc} = \frac{ab \sin C}{abc}
\]

Divide by \( abc \).

\[
\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}
\]

This final relationship is called the law of sines. If three nonzero numbers are equal, then their reciprocals are equal. So you can write the law of sines in another algebraic form:

\[
\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}
\]

PROPERTY: The Law of Sines

In \( \triangle ABC \),

\[
\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}
\]

Verbal: Within any given triangle, the ratio of the sine of an angle to the length of the side opposite that angle is constant.

Because of the different combinations of sides and angles for any given triangle, it is convenient to revive some terminology from geometry. The initials SAS stand for “side, angle, side.” This means that as you go around the perimeter of the triangle, you are given the length of a side, the measure of an angle, and the length of a side, in that order. SAS is equivalent to knowing two sides and the included angle, the same information used in the law of cosines and in the area formula. Similar meanings are attached to ASA, AAS, SSA, and SSS.

Although the law of sines is an easy and safe way to find side lengths, students must be careful when using it to find angle measures. Problem 11 illustrates the risks involved in using the law of sines to find angle measures resulting from the fact that there are two angles in the interval [0, 180°] with a given sine value. Be sure to discuss this problem in class. Exploration 9-4a presents another way to approach this problem.

The inverse sine function always gives the first-quadrant angle. To find the correct answer, students must consider the general solution for arcsine.

You might present the tables on pages 456–457 to summarize what students have learned so far about finding unknown measures in triangles. A blackline master is available in the Instructor’s Resource Book.
Given AAS, Find the Other Sides

Example 1 shows you how to calculate two side lengths given the third side and two angles.

In \( \triangle ABC \), \( b = 64^\circ \), \( C = 38^\circ \), and \( 9 \text{ ft.} \) Find the lengths of sides \( a \) and \( c \).

First, draw a picture, as in Figure 9-4b.

Because you know the angle opposite side \( c \) but not the angle opposite side \( a \), it’s easier to start with finding the length of side \( c \).

\[
\frac{c}{\sin 38^\circ} = \frac{9}{\sin 64^\circ}
\]

Use the law of sines. Put the unknown in the numerator on the left side.

\[
c = \frac{9 \sin 38^\circ}{\sin 64^\circ} = 6.1648... \text{ ft}
\]

Multiply both sides by \( \sin 38^\circ \) to isolate \( c \) on the left.

To find \( a \) by the law of sines, you need the measure of \( A \), the opposite angle.

\[
A = 180^\circ - (38^\circ + 64^\circ) = 78^\circ
\]

The sum of the interior angles in a triangle is 180°.

Use the appropriate parts of the law of sines with \( a \) in the numerator.

\[
\frac{a}{\sin 78^\circ} = \frac{9}{\sin 64^\circ}
\]

\[
a = \frac{9 \sin 78^\circ}{\sin 64^\circ} = 9.7946... \text{ ft}
\]

\[\therefore a = 9.79 \text{ ft and } c = 6.16 \text{ ft}\]

Given ASA, Find the Other Sides

Example 2 shows you how to calculate side lengths if the given side is included between the two given angles.

In \( \triangle ABC \), \( a = 8 \text{ m} \), \( B = 64^\circ \), and \( C = 38^\circ \). Find the lengths of sides \( b \) and \( c \).

First, draw a picture (Figure 9-4c). The picture reveals that in this case you do not know the angle opposite the given side. So you calculate this angle measure first. From there on, it is a familiar problem, similar to Example 1.

\[
a = 8 \text{ m}
\]

\[
\frac{a}{\sin 78^\circ} = \frac{8}{\sin 64^\circ}
\]

\[
b = \frac{8 \sin 64^\circ}{\sin 78^\circ} = 7.3509... \text{ m}
\]

### What You Are Given

<table>
<thead>
<tr>
<th>What You Want to Find</th>
<th>Law to Apply</th>
</tr>
</thead>
<tbody>
<tr>
<td>Three sides (SSS)</td>
<td>An unknown angle</td>
</tr>
<tr>
<td>Two sides and an included angle (SAS)</td>
<td>The unknown side</td>
</tr>
<tr>
<td>Two angles and an included side (ASA)</td>
<td>An unknown side</td>
</tr>
</tbody>
</table>
The Law of Sines for Angles

You can use the law of sines to find an unknown angle of a triangle. However, you must be careful because there are two values of the inverse sine relation between 0° and 180°, either of which could be the answer. For instance, arcsin 0.8 = 53.1301° or 126.8698°; both could be angles of a triangle. Problem 11 shows you what to do in this situation.

Problem Set 9-4

<table>
<thead>
<tr>
<th>Reading Analysis</th>
</tr>
</thead>
<tbody>
<tr>
<td>From what you have read in this section, what do you consider to be the main idea? Based on the verbal statement of the law of sines, why is it necessary to know at least one angle in the triangle to use the law? In the solution to Example 1, why is it advisable to put the unknown side length in the numerator on the left side of the equation? Why can you be led to an incorrect answer if you try to use the law of sines to find an angle measure?</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Quick Review</th>
</tr>
</thead>
<tbody>
<tr>
<td>Q1. State the law of cosines for ( \triangle PAF ) involving angle ( P ).</td>
</tr>
<tr>
<td>Q2. State the formula for the area of ( \triangle PAF ) involving angle ( P ).</td>
</tr>
<tr>
<td>Q3. Write two values of ( \theta = \arcsin 0.5 ) that lie between 0° and 180°.</td>
</tr>
<tr>
<td>Q4. If ( \sin \theta = 0.3726 \ldots ), then ( \sin(-\theta) = \ldots ).</td>
</tr>
<tr>
<td>Q5. ( \cos \frac{\pi}{6} = \ldots )</td>
</tr>
<tr>
<td>A. ( \frac{1}{\sqrt{3}} )</td>
</tr>
<tr>
<td>B. ( \frac{1}{2} )</td>
</tr>
<tr>
<td>C. ( \frac{2}{\sqrt{3}} )</td>
</tr>
<tr>
<td>D. ( \frac{\sqrt{3}}{2} )</td>
</tr>
<tr>
<td>E. ( \sqrt{3} )</td>
</tr>
<tr>
<td>Q6. A(n) ( \ldots ) ( \ldots ) triangle has no equal sides and no equal angles.</td>
</tr>
<tr>
<td>Q7. A(n) ( \ldots ) ( \ldots ) triangle has no right angle.</td>
</tr>
<tr>
<td>Q8. State the Pythagorean property for cosine and sine.</td>
</tr>
<tr>
<td>Q9. ( \cos 2x = \cos(x + x) = \ldots ) in terms of cosines and sines of ( x ).</td>
</tr>
<tr>
<td>Q10. The amplitude of the sinusoid ( y = 3 + 4 \cos(5x - 6) ) is ( \ldots ).</td>
</tr>
</tbody>
</table>

1. In \( \triangle ABC, A = 52°, B = 31°, \) and \( a = 8 \) cm. Find the lengths of side \( b \) and side \( c \). |
2. In \( \triangle PQR, P = 13°, Q = 133°, \) and \( q = 9 \) in. Find the lengths of side \( p \) and side \( r \). |
3. In \( \triangle AHS, A = 27°, H = 109°, \) and \( a = 120 \) yd. Find the lengths of side \( h \) and side \( s \). |
4. In \( \triangle BIG, B = 2°, I = 79°, \) and \( b = 20 \) km. Find the lengths of side \( i \) and side \( g \). |
5. In \( \triangle PAF, P = 28°, f = 6 \) m, and \( A = 117° \). Find the lengths of side \( a \) and side \( p \). |
6. In \( \triangle JAW, J = 48°, a = 5 \) ft, and \( W = 73° \). Find the lengths of side \( j \) and side \( w \). |
7. In \( \triangle ALP, A = 85°, p = 30 \) ft, and \( L = 87° \). Find the lengths of side \( a \) and side \( l \). |
8. In \( \triangle LOW, L = 2°, o = 500 \) m, and \( W = 3° \). Find the lengths of side \( l \) and side \( w \).

<table>
<thead>
<tr>
<th>Problem Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>Problems 1–8 are fairly straightforward and are similar to Examples 1 and 2.</td>
</tr>
<tr>
<td>Students can use a system of equations on a CAS as described in the CAS Suggestions to solve Problems 1–8. This method enables students to obtain the values of both side lengths using a single command.</td>
</tr>
<tr>
<td>1. ( b = 5.23 ) cm; ( c = 10.08 ) cm</td>
</tr>
<tr>
<td>2. ( p = 2.77 ) in.; ( r = 6.88 ) in.</td>
</tr>
<tr>
<td>3. ( h = 249.92 ) yd; ( s = 183.61 ) yd</td>
</tr>
<tr>
<td>4. ( i = 562.55 ) km; ( g = 566.02 ) km</td>
</tr>
<tr>
<td>5. ( a = 9.32 ) m; ( p = 4.91 ) m</td>
</tr>
<tr>
<td>6. ( j = 4.33 ) ft; ( w = 5.58 ) ft</td>
</tr>
<tr>
<td>7. ( a = 248.47 ) ft; ( l = 215.26 ) ft</td>
</tr>
<tr>
<td>8. ( l = 200.21 ) m; ( w = 300.24 ) m</td>
</tr>
</tbody>
</table>
Problem Notes (continued)

In Problem 9, students will need to recognize that the side of length 1000 is opposite the largest angle, and the other two sides each must be less than 1000.

9a. \( x = 679.4530\ldots \text{ m} ; y = 861.1306\ldots \text{ m} \)
9b. $67,220.70
9c. $105,421.05 over \( y \) and $38,200.35 over \( x \).

10a. \( = 399 \) ft
10b. \( = 1125 \) ft
10c. It is faster to retrace the original route.

Problem 11 is well worth spending time on. It helps students understand the pitfalls of using the law of sines to find angle measures. Exploration 9-4a covers the same content.

11a. \( A = 33.1229\ldots^\circ \)
11b. \( C = 51.3178\ldots^\circ \)
11c. \( C = 128.6821\ldots^\circ \)
11d. This is the complement of \( 51.3178\ldots^\circ \) and one of the general values of \( \arccos \left( \frac{10}{\sin A} \right) \).

11e. The principal values of \( \arccos x \) go from \( 0^\circ \) to \( 180^\circ \); a negative argument will give an obtuse angle and a positive argument will give an acute angle, always the actual angle in the triangle. But the principal values of \( \arcsin x \) go from \( -90^\circ \) to \( 90^\circ \); a negative argument will never happen in a triangle problem, but a positive argument will only give an acute angle, whereas the actual angle in the triangle may be the obtuse complement of the acute angle.

12. The measured value should be within 0.1 of 5.3208\ldots cm.

13. Answers will vary.

14. \( A = \frac{1}{2} x y \sin Z = \frac{1}{2} y z \sin X = \frac{1}{2} x z \sin Y \)
\[
\begin{align*}
\text{So } \frac{1}{2} x y \sin Z &= \frac{1}{2} y z \sin X, \\
x \sin Z &= z \sin X, \\
\frac{x}{\sin X} &= \frac{z}{\sin Z}, \text{ and similarly,} \\
\frac{x}{\sin X} &= \frac{y}{\sin Y}. 
\end{align*}
\]

11. Law of Sines for Angles Problem: You can use the law of sines to find an unknown angle measure, but the technique is risky. Suppose that \( \triangle ABC \) has sides 4 cm, 7 cm, and 10 cm, as shown in Figure 9-4f.

a. Use the law of cosines to find the measure of angle \( A \).

b. Use the answer to part a (don't round off) and the law of sines to find the measure of angle \( C \).

c. Find the measure of angle \( C \) again, using the law of cosines and the given side lengths.

d. Your answers to parts \( b \) and \( c \) probably do not agree. Show that you can get the correct answer from your work with the law of sines in part \( b \) by considering the general solution for \( \arcsin \).

e. Why is it dangerous to use the law of sines to find an angle measure but not dangerous to use the law of cosines?

12. Accurate Drawing Problem: Using computer software such as The Geometer's Sketchpad, or using a ruler and protractor with pencil and paper, construct a triangle with base 10.0 cm and base angles 40° and 30°. Measure the length of the side opposite the 30° angle. Then calculate its length using the law of sines. Your measured value should be within \( \pm 0.1 \) cm of the calculated value.


14. Algebraic Derivation of the Law of Sines Problem: Derive the law of sines algebraically. If you cannot do it from memory, consult the text long enough to get started. Then try finishing on your own.

See page 1019 for answers to CAS Problems 1 and 2.

458 Chapter 9: Triangle Trigonometry
9-5 The Ambiguous Case

From one end of a long segment, you draw an 80-cm segment at a 26° angle. From the other end of the 80-cm segment, you draw a 50-cm segment, completing a triangle. Figure 9-5a shows the two possible triangles you might create.

Figure 9-5a

As you go around the perimeter of the triangle in Figure 9-5b, the given information is a side, another side, and an angle (SSA). Because there are two possible triangles that have these specifications, SSA is called the ambiguous case.

Objective

Given two sides and a non-included angle, calculate the possible lengths of the third side.

Example 1

In \( \triangle XYZ \), \( z = 50 \text{ cm} \), \( x = 80 \text{ cm} \), and \( X = 26^\circ \), as in Figure 9-5a. Find the possible lengths of side \( y \).

Solution

Sketch a triangle and label the given sides and angle (Figure 9-5c).

Exploration Notes

Exploration 9-5a allows students to investigate the SSA case by measuring and drawing. You could use this activity as a follow-up to Example 1 to reinforce the concept of ambiguity and the use of the quadratic formula. Or you could complete it as a whole-class activity in place of Example 1. Allow 20 minutes for this exploration.

Exploration 9-5b (inspired by Chris Sollars) shows students a real-world situation involving the ambiguous case for analyzing the position of a golf ball. Allow 20 minutes for this exploration.
Section Notes (continued)

The computations involved in the quadratic formula can be done easily on a grapher.

If students do not have a quadratic formula program in their graphers, you might suggest that they write or download one so that they can solve the problems efficiently. Using the quadratic formula is preferable to using the solver feature because it is easy to miss one of the solutions with the solver feature.

It is worthwhile to go through the steps of the quadratic formula for the cases illustrated in Figures 9-5d and 9-5e so that students can see how the solutions relate to the figures.

The SSA diagrams are always drawn a particular way that makes it easier for students to see whether or not two triangles are a possibility. The given angle is placed on the lower left with one side of the angle on the horizontal. Move clockwise from that vertex along the given side that is not opposite the known angle. This takes you to a vertex that acts like a “pivot” point for the side opposite the given angle. Sketch this side remembering that the side will swing out (away from the given angle) or possibly swing in (toward the given angle). See Figure 9-5a. Encourage your students to draw their figures the same way.

Have a volunteer draw $\triangle ABC$ on the board or overhead with $\angle A = 30^\circ$, $b = 10$, and $a = 5$. Hopefully, students will see that the triangle is a right triangle, because 5 is half of 10 and the triangle has a $30^\circ$ angle.

Next, ask a volunteer to draw $\triangle ABC$ with $\angle A = 30^\circ$, $b = 10$, and $a = 4$. (The student can use $\angle A$ and side $b$ from the previous drawing.) Students should observe that side $a$ is too short and thus that no triangle meets the conditions.

Next, ask a volunteer to draw $\triangle ABC$ with $\angle A = 30^\circ$, $b = 10$, and $a = 6$. There are two possible triangles in this case, because side $a$ can “swing off” point $C$ toward $A$ or away from $A$. If students don’t notice that two triangles can be drawn, ask them if it is possible to draw a second triangle meeting the conditions.

Finally, ask a volunteer to draw $\triangle ABC$ with $\angle A = 30^\circ$, $b = 10$, and $a = 11$. In this case, side $a$ can only “swing out” away from $\angle A$, so there is only one triangle that meets these conditions.

Using the law of sines to find $y$ would require several steps. Here is a shorter method, using the law of cosines.

\[
50^2 = y^2 + 80^2 - 2 \cdot y \cdot 80 \cdot \cos 26^\circ
\]

Write the law of cosines for the known angle, $X = 26^\circ$.

This is a quadratic equation in the variable $y$. You can solve it using the quadratic formula.

\[
y^2 - (160 \cos 26^\circ) y + 6400 - 2500 = 0
\]

Make one side equal zero.

\[
y^2 + (-160 \cos 26^\circ) y + 3900 = 0
\]

Get the form $ay^2 + by + c = 0$.

\[
y = \frac{160 \cos 26^\circ \pm \sqrt{(-160 \cos 26^\circ)^2 - 4 \cdot 1 \cdot 3900}}{2 \cdot 1}
\]

Use the quadratic formula: $y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

\[
y = 107.5422... \text{ or } 36.2648...
\]

You may be surprised if you use different lengths for side $x$ in Example 1. Figures 9-5d and 9-5e show this side as 90 cm and 30 cm, respectively, instead of 50 cm. In the first case, there is only one possible triangle. In the second case, there is none.

The quadratic formula technique of Example 1 detects both of these results. For 30 cm, the discriminant, $b^2 - 4ac$, equals $-1319.3331...$, meaning there are no real solutions to the equation and thus no triangle. For 90 cm,

\[
y = 154.7896... \text{ or } -10.9826...
\]

Although $-10.9826...$ cannot be a side measure of a triangle, it does equal the displacement (the directed distance) to the point where the arc would cut the starting segment if this segment were extended in the other direction.

**Differentiating Instruction**

- Remind students that two triangles cannot be proved congruent by SSA, and use Figure 9-5b to illustrate the meaning of ambiguous case.
- Go over Example 1 and the explanatory text on page 460. Check carefully for understanding.
- Students may need support with the language in Problems 9 and 14.
Problem Set 9-5

Reading Analysis

From what you have read in this section, what do you consider to be the main idea? Sketch a triangle with two given sides and a given non-included angle that illustrates that there can be two different triangles with the same given information. How can the law of cosines be applied in the ambiguous case to find both possible lengths of the third side with the same computation?

Quick Review

Problems Q1–Q6 refer to the triangle in Figure 9-5f.

Figure 9-5f

Q1. The initials SAS stand for ___.
Q2. Find the length of the third side.
Q3. What method did you use in Problem Q2?
Q4. Find the area of this triangle.
Q5. The largest angle in this triangle is opposite the ___ side.
Q6. The sum of the angle measures in this triangle is ___.
Q7. Find the amplitude of the sinusoid

\[ y = 4 \cos x + 3 \sin x \]
Q8. The period of the circular function

\[ y = 3 + 7 \cos \frac{\pi}{6}(x - 1) \] is

A. 16 B. 8 C. \( \frac{\pi}{6} \) D. 7 E. 3
Q9. The value of the inverse circular function

\[ x = \sin^{-1} 0.5 \] is ___.
Q10. A value of the inverse circular relation

\[ x = \arcsin 0.5 \] between \( \frac{\pi}{6} \) and 2\( \pi \) is ___.

For Problems 1–8, find the possible lengths of the indicated side.

1. In \( \triangle ABC \), \( B = 34^\circ \), \( a = 4 \text{ cm} \), and \( b = 3 \text{ cm} \). Find \( c \).

2. In \( \triangle XYZ \), \( X = 13^\circ \), \( x = 12 \text{ ft} \), and \( y = 5 \text{ ft} \). Find \( z \).

3. In \( \triangle ABC \), \( B = 34^\circ \), \( a = 4 \text{ cm} \), and \( b = 5 \text{ cm} \). Find \( c \).

4. In \( \triangle XYZ \), \( X = 13^\circ \), \( x = 12 \text{ ft} \), and \( y = 15 \text{ ft} \). Find \( z \).

5. In \( \triangle ABC \), \( B = 34^\circ \), \( a = 4 \text{ cm} \), and \( b = 2 \text{ cm} \). Find \( c \).

6. In \( \triangle XYZ \), \( X = 13^\circ \), \( x = 12 \text{ ft} \), and \( y = 60 \text{ ft} \). Find \( z \).

7. In \( \triangle RST \), \( R = 130^\circ \), \( r = 20 \text{ in.} \), and \( t = 16 \text{ in.} \). Find \( s \).

8. In \( \triangle OBT \), \( O = 170^\circ \), \( o = 19 \text{ m} \), and \( t = 11 \text{ m} \). Find \( b \).

9. Radio Station Problem: Radio station KROK plans to broadcast rock music to people on the beach near Ocean City (O.C. in Figure 9-5g). Measurements show that Ocean City is 20 mi from KROK, at an angle 50\(^\circ\) north of west. KROK’s broadcast range is 30 mi.

a. Use the law of cosines to calculate how far along the beach to the east of Ocean City people can hear KROK.

b. There are two answers to part a. Show that both answers have meaning in the real world.

c. KROK plans to broadcast only in an angle between a line from the station through Ocean City and a line from the station through the point on the beach farthest to the east of Ocean City that people can hear the station. What is the measure of this angle?

Technology Notes

Exploration 9-5a in the Instructor’s Resource Book asks students to investigate the question of whether knowing two side lengths and a non-included angle determines a triangle. Sketchpad can be a useful tool in creating constructions.

Exploration 9-5b in the Instructor’s Resource Book demonstrates the ambiguity of SSA in the context of a golf game. Sketchpad can be useful for constructing a model.


Section 9-5: The Ambiguous Case

PROBLEM NOTES

Supplementary problems for this section are available at www.keypress.com/keyonline.

Q1. Side-Angle-Side = 4.57
Q2. The law of cosines
Q4. = 8.62
Q5. Longest
Q6. 180°
Q7. 5
Q8. A
Q9. \( \frac{\pi}{6} \)
Q10. \( \frac{5\pi}{6} \)

Problems 1–8 are straightforward problems in which students must find the unknown side length in SSA situations. Note that when there is only one possible obtuse triangle, the law of cosines produces a positive solution and a negative solution. The negative solution has a geometric meaning as a directed distance.

1. \( c = 5.32... \text{ cm or 1.32... cm} \)
2. \( z = 16.82 \text{ ft} \)
3. \( c = 7.79 \text{ cm} \)
4. \( z = 26.13 \text{ ft or 3.10 ft} \)
5. No solution.
6. No solution.
7. = 5.52 in.
8. \( b = 8.07 \text{ m} \)
9a. = 38.65 mi.
9b. The other answer is = -12.94 mi.
This means 12.94 miles to the west of Ocean City.
9c. 99.29°

Problems 9–13 require students to use the law of sines to find missing angles in the SSA case. Students must determine beforehand whether one or two triangles meet the given conditions. Remind students to exercise caution when using the law of sines to find angles. Considering the general solution of arcsine gives two possible angles (the \( \sin^{-1} \) value and its supplement). Students must determine whether one or both angles satisfy the problem.
Problem Notes (continued)

10.  $C \approx 23.00^\circ$ or $157.00^\circ$
11.  $S \approx 141.18^\circ$
12.  $Z \approx 43.15^\circ$
13.  $G \approx 57.75^\circ$

Problem 14 offers a nice summary of the six different possibilities SSA problems present. This is a good problem to discuss in class if you do not want to assign it for homework.

14a.  $x = y \sin X < y$
14b.  $x < y \sin X < y$
14c.  $y \sin X < y < x$
14d.  $y \sin X < x < y$
14e.  $y \sin X < y < x$
14f.  $x < y \sin X < y$

For Problems 10–13, use the law of sines to find the indicated angle measure. Determine beforehand whether there are two possible angles or just one.

10. In $\triangle ABC$, $A = 19^\circ$, $a = 25$ mi, and $c = 30$ mi. Find $C$.
11. In $\triangle HSC$, $H = 28^\circ$, $h = 50$ mm, and $c = 20$ mm. Find $S$.
12. In $\triangle XYZ$, $X = 58^\circ$, $x = 9.3$ cm, and $z = 7.5$ cm. Find $Z$.
13. In $\triangle BIG$, $B = 110^\circ$, $b = 1000$ yd, and $g = 900$ yd. Find $G$.
14. Six SSA Possibilities Problem: Parts a through f show six possibilities of $\triangle XYZ$ if angle $X$ and sides $x$ and $y$ are given. For each case, explain the relationship among $x$, $y$, and the quantity $y \sin X$

Additional CAS Problems

1. Suppose the sides of a triangle form an arithmetic sequence. If one of the angles is right and the lengths of all sides are integers, what are the lengths of the sides?
2. In $\triangle XYZ$, $y = 10$, the measure of angle $X$ is $30^\circ$, and side $z$ is “$a$” units longer than side $x$.
   a. Is this ever an ambiguous triangle?
   b. Under what conditions for “$a$” is there only one triangle possible?

See page 1019 for answers to CAS Problems 1 and 2.

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**9-6 Vector Addition**

Suppose you start at the corner of a room and walk 10 ft at an angle of 70° to one of the walls (Figure 9-6a). Then you turn 80° clockwise and walk another 7 ft. If you had walked straight from the corner to your stopping point, how far and in what direction would you have walked?

The two motions described are called displacements. They are vector quantities that have both magnitude (size) and direction (angle). Vector quantities are represented by directed line segments called vectors. A quantity such as distance, time, or volume that has no direction is called a scalar quantity.

**Objective**

Given two vectors, add them to find the resultant vector.

In this exploration you will use the properties of triangles to add vectors.

### Exploration 9-6: Sum of Two Displacement Vectors

1. The figure shows two vectors starting from the origin. One ends at the point (4, 7), and the other ends at the point (5, 3). Copy the figure on graph paper and translate one of the two vectors so that the beginning of the translated vector is at the end of the other vector. Then draw the resultant vector—the sum of the two vectors.

2. Calculate the length of the resultant vector in Problem 1 and the angle it makes with the x-axis.

3. The two given vectors and the resultant vector form a triangle. Calculate the measure of the largest angle in this triangle.

4. Calculate the measure of the angle between the two vectors when they are placed tail-to-tail, as they were given in Problem 1.

5. In Problem 1, you translated one of the vectors. Show on your copy of the figure that you would have gotten the same resultant vector if you had translated the other vector. Use a different color than you used in Problem 1.

**Exploration Notes**

*Exploration 9-6* begins by having students use triangle properties to find the sum of two vectors. Students are then led to discover an “easier” way to find the sum, using components. This is a good preview of the ideas in this section. Allow students 15 minutes to complete the exploration.

See page 468 for notes on additional explorations.

2. \(|\vec{r}| = \sqrt{181} = 13.5; \theta = 48.0°\)

3. \(\theta = 150.7086° \approx 150.7°\)

4. \(29.3°\)

7. The \(\vec{x}\)-component of the resultant vector is the sum of the \(\vec{x}\)-components of the given vectors, and the \(\vec{y}\)-component is the sum of the \(\vec{y}\)-components.

8. Answers will vary.

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**Section 9-6**

**PLANNING**

**Class Time**

2 days

**Homework Assignment**

*Day 1*: RA, Q1–Q10, Problems 1, 3, 5, 7, 9

*Day 2*: Problems 13–15, 17, 19, 21, 22, 24

**Teaching Resources**

Exploration 9-6: Sum of Two Displacement Vectors

Exploration 9-6a: Navigation Vectors

Supplementary Problems

**Technology Resources**

- Exploration 9-6: Sum of Two Displacement Vectors

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**TEACHING**

**Important Terms and Concepts**

Displacement vector

Vector quantity

Magnitude of a vector

Direction of a vector

Scalar quantity

Head of a vector

Tail of a vector

Absolute value

Equal vectors

Translate a vector

Resultant vector

Unit vector

Components of a vector

Resolving a vector

Bearing

Opposite of a vector

Commutative

Associative

Zero vector

Closed

See page 1019–1020 for answers to Exploration Problems 1, 5, and 6.

---

Section 9-6: Vector Addition 463
**Section Notes**

This section introduces students to vectors in a geometrical context that lays the foundation for the work with three-dimensional vectors in Chapter 12. This section contains problems that are applications of the law of cosines, and it also introduces vector components to solve the problems in a more direct manner. Theoretical concepts such as commutativity and associativity are touched on in the problem set (Problems 22–26). It is recommended that you spend two days on this section. Cover Examples 1 and 2 on the first day and the remaining examples on the second day.

This section introduces many new terms. Encourage students to use correct terminology when discussing and writing about vectors.

Emphasize to students that a vector has a direction and a magnitude but does not have a specific location. This makes it possible to translate vectors in order to add or subtract them. (The only exception to the rule that a vector does not have a location is the position vector which has its tail at the origin.)

Example 1 shows how to use triangle trigonometry to find the displacement that results from combining two motions. The resultant displacement is the vector from the beginning of the vector representing the first motion to the end of the vector representing the second motion. This resultant vector is the sum of the two motion vectors. Having a student act out the scenario in this example might help students gain insight into the resultant vector concept.

Example 1 leads to the definition of the sum of the two vectors. Discuss the definition with students and illustrate it with a drawing.

**EXPLORATION, continued**

6. The vectors in Problem 1 have components in the \( x \)-direction and in the \( y \)-direction. These components are a horizontal vector and a vertical vector that can be added together to equal the given vector. On your graph from Problem 1, show how the components of the longer vector can be added to give that vector.

7. Give an easy way to get the components of the resultant vector of the two given vectors in Problem 1.

8. What did you learn as a result of doing this exploration that you did not know before?

**DEFINITION: Vector**

A vector, \( \vec{v} \), is a directed line segment.

The absolute value, or magnitude, of a vector, \( |\vec{v}| \), is a scalar quantity equal to its length.

Two vectors are equal if and only if they have the same magnitude and the same direction.

**EXAMPLE 1**

**SOLUTION**

You start at the corner of a room and walk as shown in Figure 9-6a. Find the displacement that results from the two motions.

Draw a diagram showing the two given vectors and the displacement that results, \( \vec{X} \) (Figure 9-6d). They form a triangle with sides 10 ft and 7 ft and included angle 100° (180° − 80°).

\[
|X|^2 = 10^2 + 7^2 - 2(10)(7) \cos 100° = 173.3107...
\]

Use the law of cosines.

\[
|X| = 13.1647... \text{ ft}
\]

Store without rounding for use later.

Translation implies that the vector is moved without changing its magnitude or direction. Remind students that any two vectors with the same magnitude and direction are equal, so translating a vector does not change its value. Example 2 shows how a tail-to-tail problem can be translated into a head-to-tail problem and then solved by triangle methods.
EXPLORATION, continued

Figure 9-6b

Figure 9-6c

Figure 9-6d

DEFINITION: Vector Addition

The sum \( \vec{a} + \vec{b} \) is the vector from the beginning of \( \vec{a} \) to the end of \( \vec{b} \) if the tail of \( \vec{b} \) is placed at the head of \( \vec{a} \).

Example 2 shows how to add two vectors that are not yet head-to-tail, using velocity vectors, for which the magnitude is the scalar speed.

A ship near the coast is going 9 knots at an angle of 130° to a current of 4 knots. What is the ship’s resultant velocity with respect to the shore?

Draw a diagram showing two vectors 9 and 4 units long, tail-to-tail, making an angle of 130° with each other, as shown in Figure 9-6e. Translate one of the vectors so that the two vectors are head-to-tail. Draw the resultant vector, \( \vec{V} \), from the beginning (tail) of the first to the end (head) of the second.

In its new position, the 4-kt vector is parallel to its original position. The 9-kt vector is a transversal cutting two parallel lines. So the angle between the vectors forming the triangle shown in Figure 9-6e is the supplement of the given 130° angle, namely, 50°. From here on the problem is like Example 1.

\[
\begin{align*}
|\vec{v}| &= 7.1217\ldots \text{ kt} \\
A &= 25.4838\ldots \\
\theta &= 130° - 25.4838\ldots = 104.5161\ldots \\
\therefore \vec{v} &= 7.1 \text{ kt at about } 104.5°
\end{align*}
\]

Discuss the airplane situation on page 466. In this example, the horizontal and vertical velocities are written as scalar multiples of \( \vec{i} \) and \( \vec{j} \), respectively, and then these horizontal and vertical vectors are added to get the resultant velocity vector.

Then explain that any vector can be written as the sum of a horizontal vector and a vertical vector. For example, if you know a plane’s speed and angle of climb, you can calculate the climb velocity and the ground velocity and then write the velocity vector as the sum of horizontal and vertical components.

The process of resolving a vector into horizontal and vertical components is illustrated in Example 3. The sum of the components is equal to the original vector. The example leads to a general property for resolving a vector into its components.

Note that some texts represent vectors as ordered pairs. The notation \( \vec{v} = x \vec{i} + y \vec{j} \) used in this text is conceptually easier for most students. It reminds them that a vector is the sum of a horizontal vector and a vertical vector.

When vectors are written in terms of their components, you can add them by adding the corresponding components. This is illustrated in Figure 9-6h.

Example 4 shows how to add two vectors by first resolving them into components. In part b, emphasize the importance of making a drawing to determine which value of arctan to choose.
Section Notes (continued)

You might also work with the class to find the sum in Example 4 by using triangle techniques. Most students will find that adding the vectors by first resolving them into components is the easier method.

Example 5 investigates a navigation problem. Make sure students understand that a bearing is measured clockwise from north rather than counterclockwise from the positive horizontal axis. When solving navigation problems, some students may be more comfortable changing the bearings to standard angle measures. The solution for Example 5 uses triangle methods, but you may want to redo it using components. (For more practice using components to solve navigation problems, see Exploration 9-6b.)

Addition is the only vector operation covered in this chapter. You may also want to show students how to subtract vectors and how to multiply a vector by a scalar, topics covered in Section 12-2.

Subtracting Vectors

Remind students that you subtract a number by adding its opposite. For example, $5 - 2 = 5 + (-2)$. In the same way, you subtract a vector by adding its opposite. The opposite of a vector is a vector with the same magnitude that points in the opposite direction. The figure illustrates a difference of two vectors.

![Diagram showing vector subtraction]

**DEFINITION:**

**Vector Subtraction**

The opposite of $b$, written $-b$, is a vector of the same magnitude as $b$ that points in the opposite direction. The difference $\vec{a} - \vec{b}$ is the sum $\vec{a} + (-\vec{b})$.

If vectors are written as the sum of components, you can subtract them by subtracting corresponding components. You can illustrate this by revisiting Example 4 and finding the difference $\vec{a} - \vec{b}$ as the sum of two components.

**Multiplying a Vector by a Scalar**

When you add the real number $x$ to itself, you get twice that number.

\[ x + x = 2x \]
Example 3 demonstrates the following property.

**PROPERTY: Components of a Vector**

If \( \vec{v} \) is a vector in the direction \( \theta \) in standard position, then

\[
\vec{v} = x \vec{i} + y \vec{j}
\]

where \( x = |\vec{v}| \cos \theta \) and \( y = |\vec{v}| \sin \theta \).

Components make it easy to add two vectors. As shown in Figure 9-6h, if \( \vec{r} \) is the resultant vector of \( \vec{a} \) and \( \vec{b} \), then the components of \( \vec{r} \) are the sums of the components of \( \vec{a} \) and \( \vec{b} \). Because the two horizontal components have the same direction, you can add them simply by adding their coefficients. The same is true for the vertical components.

![Diagram](image)

**EXAMPLE 4**

Vector \( \vec{a} \) has magnitude 5 at 70°, and \( \vec{b} \) has magnitude 6 at 25° (Figure 9-6h). Find the resultant vector, \( \vec{r} \), as

- a. The sum of two components
- b. A magnitude and a direction angle

**SOLUTION**

a. \( \vec{r} = \vec{a} + \vec{b} \)

\[
= (5 \cos 70°) \vec{i} + (5 \sin 70°) \vec{j} + (6 \cos 25°) \vec{i} + (6 \sin 25°) \vec{j}
\]

Write the components.

\[
= 7.1479\vec{i} + 7.2341\vec{j}
\]

Combine like terms.

\[
= 7.15\vec{i} + 7.23\vec{j}
\]

Round the final answer.

b. \( |\vec{r}| = \sqrt{(7.1479)^2 + (7.2341)^2} = 10.1698... \)

By the Pythagorean theorem.

\[
\theta = \arctan \frac{7.2341}{7.1479} = 45.345..° + 180°n = 45.345..° + 0°
\]

Pick \( n = 0 \).

\( \therefore \vec{r} \approx 10.17 \) at 45.34°

Round the final answer.

It is reasonable to say that when you add a vector to itself, you get twice that vector.

\( \vec{a} + \vec{a} = 2\vec{a} \)

When you multiply a real number \( x \) by \(-1\), you get the opposite of that number.

\( -1 \cdot x = -x \)

In a similar way, when you multiply a vector by \(-1\), you get the opposite of that vector.

\( -1 \cdot \vec{a} = -\vec{a} \)

This reasoning leads to the definition of the product of a scalar (that is, a real number) and a vector.

**DEFINITION: Product of a Scalar and a Vector**

The product \( x\vec{a} \) is a vector in the direction of \( \vec{a} \) if \( x \) is positive and in the direction of \( -\vec{a} \) if \( x \) is negative. The magnitude of the product is the magnitude of \( \vec{a} \) times the absolute value of \( x \).

**Differentiating Instruction**

- Check to see whether any of your students have learned a different notation for vectors than the one used in the text. If so, allow them to use either notation.
- ELL students should work on *Exploration 9-6* in pairs; the language is more complicated than it appears.
- Example 2 uses language that is probably unfamiliar to many students. Check carefully for understanding.
- Monitor students’ understanding of the new material on vector addition by components and components of a vector, on pages 466–467.
- The navigation problems introduced on page 468 present several challenges for students. The position of bearing 0° is different from a 0° angle in standard position. Also, the vocabulary in navigation problems is not commonly used in conversation. You might work through a homework problem, such as *Problem 17*, as another example. Emphasize the importance of drawing a diagram to represent the situation.
- The language in the problem set will present challenges for ELL students. Have ELL students work in pairs, and give them a shorter assignment so that they have time to work through new vocabulary.
**Additional Exploration Notes**

*Exploration 9-6a* requires students to use components to solve a navigation problem. You may need to remind students about the difference between a standard-position angle and a bearing and how to get from one to the other. Allow students 20 minutes to complete this activity.

**Technology Notes**

*Exploration 9-6* guides students through the addition of vectors, using properties of triangles. This exploration can be done with the aid of Sketchpad.

**CAS Suggestions**

Vectors are noted on a TI-Nspire CAS with square brackets. As noted in the text, their components can be displayed in either component form or using magnitude and direction. The two parts of each vector’s definition are separated by commas even though the output uses spaces. Finally, angles are indicated using the angle symbol.

The first vector from Example 1 can be written as $\vec{a} = [10 \text{ ft}, \angle 70^\circ]$ and the second as $\vec{b} = [7 \text{ ft}, \angle -10^\circ]$. (Use corresponding angles to transfer the $70^\circ$-angle at the base of the first vector to its tip and note the $80^\circ$ angle is actually $-10^\circ$ from horizontal.) With this setup, vector addition gives the magnitude and direction of $\vec{x} = \vec{a} + \vec{b}$. The figure confirms Example 1.

Using CAS vector notation, the current in Example 2 is $[4 \text{ knot}, \angle 0^\circ]$ and the boat is $[9 \text{ knot}, \angle 130^\circ]$. The vector sum is given in the second line of the next figure with the conversion of the magnitude back to knots in line 3.

**Navigation Problems**

A bearing, an angle measured clockwise from north, is used universally by navigators for a velocity or a displacement vector. Figure 9-6i shows a bearing of 250°.

**EXAMPLE 5**

Victoria walks 90 m due south (bearing 180°), then turns and walks 40 m more along a bearing of 250° (Figure 9-6j).

a. Find her resultant displacement vector from the starting point.

b. What is the starting point’s bearing from the place where Victoria stops?

**SOLUTION**

To find the bearing, first calculate the measure of angle $\beta$ in the resulting triangle.

$$\cos \beta = \frac{90^2 + (110.2839...)^2 - 40^2}{2(90)(110.2839...)}$$

$$\beta = 99.9272^\circ$$

Bearing = $180^\circ + 99.9272^\circ = 279.9272^\circ$ See Figure 9-6j.

$\vec{r} = 110.3$ m at a bearing of 199.9°
b. The bearing from the ending point to the starting point is the opposite of the bearing from the starting point to the ending point. To find the opposite, add 180° to the original bearing.

\[
\text{Bearing } = 199.9272...° + 180° = 379.9272...°
\]

Because this bearing is greater than 360°, find a coterminal angle by subtracting 360°.

\[
\text{Bearing } = 379.9272...° - 360° = 19.9°
\]

**Problem Set 9-6**

**Reading Analysis**

From what you have read in this section, what do you consider to be the main idea? What is the difference between a vector quantity and a scalar quantity? What is the difference between a vector and a vector quantity? What is the magnitude of a vector? How can they be used to write the components of a vector in an xy-coordinate system?

**Quick Review**

Q1. \( \cos 90° = \)
   A. 1  B. 0  C. \(-1\)  D. \(\frac{1}{2}\)  E. \(\sqrt{3}\)

Q2. \( \tan \frac{\pi}{4} = \)
   A. 1  B. 0  C. \(-1\)  D. \(\frac{1}{2}\)  E. \(\sqrt{3}\)

Q3. In \( \triangle FED \), the sine of the cosine states that \( f^2 = ? \).

Q4. A triangle has sides 5 ft and 8 ft and included angle 30°. What is the area of the triangle?

Q5. For \( \triangle MNO \), \( \sin M = 0.12 \), \( \sin N = 0.3 \), and \( \text{side } m = 24 \text{ cm} \). How long is side \( n \)?

Q6. Finding the equation of a sinusoid that is combined to form a graph is called \( ? \).

Q7. If \( \sin \theta = \frac{1}{2} \) and angle \( \theta \) is in Quadrant II, what is \( \cos \theta \)?

Q8. If \( \theta = \csc^{-1}\left(\frac{11}{2}\right) \), then \( \sin^{-1}\left(\frac{1}{2}\right) \).

Q9. The equation \( y = 3 \cdot 5^x \) represents a particular \( ? \) function.

Q10. What transformation is applied to \( f(x) \) to get \( g(x) = f(3x) \)?

For Problems 1–4, translate one vector so that the two vectors are head-to-tail, and then use appropriate triangle trigonometry to find \( |\vec{a} + \vec{b}| \) and the angle the resultant vector makes with \( \vec{a} \) (Figure 9-6k).

\[
\vec{a} \quad \theta \\
\vec{b}
\]

**Figure 9-6k**

Q1. \( |\vec{a}| = 7 \text{ cm} \), \( |\vec{b}| = 11 \text{ cm} \), and \( \theta = 73° \)
Q2. \( |\vec{a}| = 8 \text{ ft} \), \( |\vec{b}| = 2 \text{ ft} \), and \( \theta = 41° \)
Q3. \( |\vec{a}| = 9 \text{ in} \), \( |\vec{b}| = 20 \text{ in} \), and \( \theta = 163° \)
Q4. \( |\vec{a}| = 10 \text{ mi} \), \( |\vec{b}| = 30 \text{ mi} \), and \( \theta = 122° \)

5. **Displacement Vector Problem**: Lucy walks on a bearing of 90° (due east) for 100 m and then on a bearing of 180° (due south) for 180 m.

a. What is her bearing from the starting point?

b. What is the starting point’s bearing from where she stops?

c. How far along the bearing in part b must Lucy walk in order to go directly back to the starting point?

To change vectors between component and magnitude-direction forms, one could change the system settings or use the conversion command: ➤F electorate changes a vector to component form, and ➤P o lar changes a vector to magnitude-direction form. The most valuable part of these two conversions is that it doesn’t matter which form the vector was originally in.

**Problem Notes**

Supplementary problems for this section are available at www.keypress.com/keyonline.

Encourage students to draw diagrams to accompany their work with vectors. Remind them that a vector must include an arrowhead to indicate direction.

Using vectors on a CAS simplifies much of the algebra required when using trigonometric functions.

Q1. B
Q2. A
Q3. \( d^2 + e^2 - 2de \cos F \)
Q4. 10 ft²
Q5. 60 cm
Q6. Harmonic analysis
Q7. 12/13
Q8. 7/11
Q9. Exponential

Q10. Horizontal dilation by a factor of \( \frac{1}{3} \)

For Problems 1–4 and 6–10, one approach would be to assume \( \vec{a} \) travels along the x-axis. Then the angle between the resultant vector and the x-axis is also the requested angle between the resultant vector and \( \vec{a} \).

1. \( |\vec{a} + \vec{b}| = 14.66 \text{ cm}; \alpha = 45.84° \)
2. \( |\vec{a} + \vec{b}| = 9.60 \text{ ft}; \alpha = 7.86° \)
3. \( |\vec{a} + \vec{b}| = 11.69 \text{ in}; \alpha = 150.00° \)
4. \( |\vec{a} + \vec{b}| = 26.12 \text{ mi}; \alpha = 103.05° \)

**Problem 5 and Problems 17–20 involve bearings rather than standard-position angles.**

5a. Lucy’s bearing is 150.9453...°

5b. The starting point’s bearing from Lucy is 330.9453...°.

5c. 205.9126... m
6. **Velocity Vector Problem**: A plane flying with an air velocity of 400 mi/h crosses the jet stream, which is blowing at 150 mi/h. The angle between the two velocity vectors is 42° (Figure 9-6l). The plane’s actual velocity with respect to the ground is the vector sum of these two velocities.

![Figure 9-6l](image)

- **a.** What is the plane’s actual velocity with respect to the ground? Why is it less than 400 mi/h + 150 mi/h?
- **b.** What angle does the plane’s ground velocity vector make with its 400-mi/h air velocity vector?

7. **Force Vector Problem**: Abe and Bill cooperate to pull a tree stump out of the ground. They think it will take a force of 350 lb to do the job. They tie ropes around the stump. Abe pulls his rope with a force of 200 lb, and Bill pulls his rope with a force of 150 lb. The force vectors make an angle of 40°, as shown in Figure 9-6m.

![Figure 9-6m](image)

- **a.** Find the magnitude of the resultant force vector and the angle the resultant vector makes with Abe’s vector.
- **b.** What false assumption about vectors did Abe and Bill make?

8. **Swimming Problem**: Suppose that you swim across a stream that has a 5-km/h current.

![Swimmer](image)

- **a.** Find your actual velocity vector if you swim perpendicular to the current at 3 km/h.
- **b.** Find your speed through the water if you swim perpendicular to the current but your resultant velocity makes an angle of 34° with the direction you are heading.
- **c.** If you swim at 3 km/h, can you make it straight across the stream? Explain.

For Problems 9–12, resolve the vector into horizontal and vertical components.

9. 

10. 

11. 

12. 

---

**Problem Notes (continued)**

6a. \(|\vec{r}| = 521.23 \text{ mi/h}\)

   The resultant could equal 400 + 150 only if the velocities were in the same direction.

6b. \(\alpha \approx 11.10^\circ\)

7a. \(|\vec{r}| = 329.3 \text{ lb; } \alpha \approx 17.02^\circ\)

7b. Abe and Bill neglected the fact that the magnitude of the sum of two vectors does not equal the sum of the magnitudes if the vectors do not point in the same direction.

8a. \(|\vec{r}| = 5.8309... \text{ km/h; } \theta = 59.0362...^\circ\) from the perpendicular

8b. 7.4128... km/h

8c. No. Any upstream component of your 3 km/h velocity can never cancel the 5 km/h downstream component of the water.

9. 6.0376... \(\vec{i} - 5.2484... \vec{j}\)

10. \(-1782.0130... \vec{i} - 907.9809... \vec{j}\)

11. \(-6.1344... \vec{i} + 14.4519... \vec{j}\)

12. 797.4644... \(\vec{i} + 306.1179... \vec{j}\)
13. **Airplane Vector Components Problem:** A jet plane flying with a velocity of 500 mi/h through the air is climbing at an angle of 35° to the horizontal (Figure 9-6n).

![Figure 9-6n](image)

(a) The magnitude of the horizontal component of the velocity vector represents the plane’s ground speed. Find this ground speed.
(b) The magnitude of the vertical component of the velocity vector represents the plane’s climb rate. How many feet per second is the plane climbing? (Recall that a mile is 5280 ft.)

14. **Baseball Vector Components Problem:** At time \( t = 0 \) s, a baseball is hit with a velocity of 150 ft/s at an angle of 25° to the horizontal (Figure 9-7a). At time \( t = 3 \) s, the ball has slowed to 100 ft/s and is going downward at an angle of 12° to the horizontal.

![Figure 9-7a](image)

(a) Find the magnitudes of the horizontal and vertical components of the velocity vector at time \( t = 0 \) s. What information do these components give you about the motion of the baseball?
(b) How fast is the baseball dropping at time \( t = 3 \) s? What mathematical quantity reveals this information?

15. If \( \vec{u} = 21 \) units at 70° and \( \vec{v} = 40 \) units at 120°, find \( \vec{u} + \vec{v} \).

(a) As a sum of two components
(b) As a magnitude and direction

16. If \( \vec{u} = 12 \) units at 60° and \( \vec{v} = 8 \) units at 310°, find \( \vec{u} + \vec{v} \).

(a) As a sum of two components
(b) As a magnitude and direction

17. A ship sails 50 mi on a bearing of 20° and then turns and sails 30 mi on a bearing of 80°. Find the resultant displacement vector as a distance and a bearing.

18. A plane flies 30 mi on a bearing of 200° and then turns and flies 40 mi on a bearing of 10°. Find the resultant displacement vector as a distance and a bearing.

19. A plane flies 200 mi/h on a bearing of 320°. The air is moving with a wind speed of 60 mi/h on a bearing of 190°. Find the plane’s resultant velocity vector (speed and bearing) by adding these two velocity vectors.

20. A scuba diver swims 100 ft/min on a bearing of 170°. The water is moving with a current of 30 ft/min on a bearing of 115°. Find the diver’s resultant velocity (speed and bearing) by adding these two velocity vectors.

21. **Spaceship Problem:** A spaceship is moving in the plane of the Sun, the Moon, and Earth. It is being acted upon by three forces (Figure 9-6p). The Sun pulls with a force of 90 newtons at 40°. The Moon pulls with a force of 50 newtons at 110°. Earth pulls with a force of 70 newtons at 230°. What is the resultant force as a sum of two components? What is the magnitude of this force? In what direction will the spaceship move as a result of these forces?

![Figure 9-6p](image)

- **Problems 15–16** can be entered as is and converted to either form as shown in the CAS suggestions.
- **Problems 15–16** can be entered as is and converted to either form as shown in the CAS suggestions.
Problem Notes (continued)

Problems 22–25 explore the properties of vector addition. In Problems 25 and 26 you may need to remind students what closure means. For the set of vectors to be closed under addition, the sum of any two vectors must be another vector.

22a. The resultant vector is the same regardless of the order in which you add the vectors.

22b. The magnitude is 0; the direction is undefined. The resultant is the vector $0\hat{i} + 0\hat{j}$, the zero vector.

22c. The resultant vector is the same regardless of the order in which you add the vectors.

23. If $a\hat{i} + b\hat{j}$ and $c\hat{i} + d\hat{j}$ are any two vectors, then $a$, $b$, $c$, and $d$ are real numbers. So $a + c$ and $b + d$ are also real numbers, because the real numbers are closed under addition. Therefore, the sum $(a + c)\hat{i} + (b + d)\hat{j}$ exists and is a vector, so the set of vectors is closed under addition. The zero vector is necessary so that the sum of any vector $a\hat{i} + b\hat{j}$ and its opposite, $-a\hat{i} + b\hat{j}$, will exist.

24. If $a\hat{i} + b\hat{j}$ is any vector, then $a$ and $b$ are real numbers. So, if $c$ is any scalar, i.e., a real number, then $ca$ and $cb$ are real numbers. So the product $ca\hat{i} + cb\hat{j}$ exists and is a vector. Therefore, the set of vectors is closed under scalar multiplication. The zero vector is necessary so that the product of any vector with the scalar 0 will exist.

25. Associativity Problem: Show that vector addition is associative by plotting on graph paper $(\vec{a} + \vec{b}) + \vec{c}$ and $\vec{a} + (\vec{b} + \vec{c})$.

26. Zero Vector Problem: Plot on graph paper the sum $\vec{a} + (-\vec{a})$. What is the magnitude of the resultant vector? Can you assign a direction to the resultant vector? Why is the resultant called the zero vector?

27. Scalar is from the Latin scūlae, meaning “ladder.”
9-7 Real-World Triangle Problems

Previously in this chapter you encountered some real-world triangle problems in connection with learning the law of cosines, the law of sines, the area formula, and Hero’s formula. You were able to tell which technique to use by the section of the chapter in which the problem appeared. In this section you will encounter such problems without having those external clues.

Objective

Given a real-world problem, identify a triangle and use the appropriate technique to calculate unknown side lengths and angle measures.

To accomplish this objective, it helps to formulate some conclusions about which method is appropriate for a given set of information. Some of these conclusions are contained in this box.

PROCEDURES: Triangle Techniques

Law of Cosines
- Usually you use it to find the length of the third side from two sides and the included angle (SAS).
- You can also use it in reverse to find an angle measure if you know three sides (SSS).
- You can use it to find both lengths of the third side in the ambiguous SSA case.
- You can’t use it if you know only one side because it involves all three sides.

Law of Sines
- Usually you use it to find a side length when you know an angle, the opposite side, and another angle (ASA or AAS).
- You can also use it to find an angle measure, but there are two values of arcsine between 0° and 180° that could be the answer.
- You can’t use it for the SSS case because you must know at least one angle.
- You can’t use it for the SAS case because the side opposite the angle is unknown.

Area Formula
- You can use it to find the area from two sides and the included angle (SAS).

Hero’s Formula
- You can use it to find the area from three sides (SSS).

Differentiating Instruction
- Have students copy the triangle techniques on page 473 into their journals and then rewrite them in their own words. Encourage them to add diagrams to clarify the descriptions.
- The Reading Analysis should be done individually and then checked for accuracy.
- Have ELL students work on the problem set in pairs. Consider shortening the assignment, and be prepared to offer support with language.
- Lead Problem 12 or 13 as a whole class activity.

Section 9-7

Class Time
2 days

Homework Assignment
Day 1: RA, Q1–Q10, Problems 1–9 odd
Day 2: Problems 11–17 odd, 18, and have students write their own problems (see Problem Notes)

Teaching Resources
Exploration 9-7a: The Ship’s Path Problem
Exploration 9-7b: Area of a Regular Polygon
Supplementary Problems

Technology Resources
- Exploration 9-7a: The Ship’s Path Problem
- Calculator Program: AREGPOLY

Section Notes

In this section, students solve a variety of triangle problems. Some real-world problems have been incorporated into earlier sections in this chapter, so two days is a reasonable amount of time to spend on this section.

Remind students that in addition to the new ideas from this chapter—the law of sines, law of cosines, and vector properties—they can use the right triangle properties they learned in Chapter 5. If a problem involves a right triangle, it is easier to use a property such as \[ \text{opposite} = \frac{\text{opposite}}{\text{hypotenuse}} \] than the law of sines or law of cosines.
The procedures box on page 473 summarizes the triangle techniques students studied in this chapter. Students should copy this information into their notebooks, along with the appropriate formulas and any pertinent material from earlier chapters. This activity will help them organize the ideas in this chapter and create a guide they can use when they solve problems.

Consider allowing each group of students to choose a problem from the problem set to present to the class, emphasizing its unique characteristics.

In addition to assigning the problems in the book, you might consider asking students to write their own problems. The problems students write themselves are often the most interesting and creative.

**Exploration Notes**

There are two explorations for this section. They can be assigned in class or used as a group project, a group quiz, or an independent homework assignment.

**Exploration 9-7a** is a real-world triangle problem that involves the ambiguous case. From a verbal description, students are to make a diagram to analyze the situation and solve the problem. Allow about 20 minutes for this activity.

**Exploration 9-7b** involves the area of regular polygons and uses the idea of a limit as the number of sides increases. This exploration shows numerically that the limit of the areas of the inscribed \( n \)-gon approaches the area of the circle \((3.141592654\ldots)\) as \( n \) increases. Students may need help in writing the program in Problem 5; they can use the program AREGPOLY, available at www.keymath.com/precalc. See the Technology Notes for more information about using this program.

---

**Problem Set 9-7**

**Reading Analysis**

From what you have read in this section, what do you consider to be the main idea? Under what condition could you not use the law of cosines for a triangle problem? Under what conditions could you not use the law of sines for a triangle problem? In each case tell why you couldn’t.

**Quick Review**

1. For \( \triangle ABC \), write the law of cosines involving angle \( B \).
2. For \( \triangle ABC \), write the law of sines involving angles \( A \) and \( C \).
3. For \( \triangle ABC \), write the area formula involving angle \( A \).
4. Sketch \( \triangle XYZ \) given \( x, y, \) and angle \( X \), showing how you can draw two possible triangles.
5. Draw a sketch showing a vector sum.
6. Draw a sketch showing the components of \( \vec{v} \).
7. Write \( \vec{a} + \vec{b} \) if \( \vec{a} = 4\vec{i} + 7\vec{j} \) and \( \vec{b} = -6\vec{i} + 8\vec{j} \).
8. \[ \cos \pi = \]
   - A. 1
   - B. 0
   - C. \(-1\)
   - D. \(\frac{1}{2}\)
   - E. \(\frac{\sqrt{3}}{2}\)
9. By the composite argument properties, \( \sin(A - B) = ? \).
10. What is the phase displacement of \( y = 7 + 6 \cos(5\theta + 37^\circ) \) with respect to the parent cosine function?

**1. Mountain Height Problem:** A surveying crew has the job of measuring the height of a mountain (Figure 9-7a). From a point on level ground they measure an angle of elevation of 21.6° to the top of the mountain. They move 507 m closer horizontally and find that the angle of elevation is now 35.8°. How high is the mountain? (You might have to calculate some other information along the way!)

**Technology Notes**

**Exploration 9-7a in the Instructor’s Resource Book** asks students to use the properties they’ve learned in this chapter to answer questions about a ship’s path. Students are asked to construct a diagram to model the ship’s movement, and this can easily be done in Sketchpad.

**Calculator Program:** AREGPOLY sets the calculator to degree mode, fixes a nine-digit output, and then displays areas of a polygon with radius 10, as the number of sides increases. For a TI-83 or TI-84, the program looks like this:

```plaintext
:Degree:Fix 9
:For(X,3,1000)
:Disp 50X*sin(360/X)
:End
```
7. Rocket Problem: An observer 2 km from the launchpad observes a rocket ascending vertically. At one instant, the angle of elevation is 21°. Five seconds later, the angle has increased to 35°.

b. What is the area of the region enclosed by the triangle?

4. Pumpkin Sale Problem: Scorpion Gulch Shelter is having a pumpkin sale for Halloween. The pumpkins will be displayed on a triangular region in the parking lot, with sides 40 ft, 70 ft, and 100 ft. Each pumpkin takes about 3 ft² of space.

a. About how many pumpkins can the shelter display?

b. Find the measure of the middle-size angle.

5. Underwater Research Lab Problem: A ship is sailing on a path that will take it directly over an occupied research lab on the ocean floor. Initially, the lab is 1000 yd from the ship on a line that makes an angle of 6° with the surface (Figure 9-7d). When the ship’s slant distance has decreased to 400 yd, the ship can contact people in the lab by underwater telephone. Find the two distances from the starting point at which the ship is at a slant distance of 400 yd from the lab.

6. Truss Problem: A builder has specifications for a triangular truss to hold up a roof. The horizontal side of the triangle will be 30 ft long. An angle at one end of this side will be 50°. The side to be constructed at the other end will be 20 ft long. Use the law of sines to find the angle measure opposite the 30-ft side. Interpret the results.

Supplementary problems for this section are available at www.keypress.com/keyonline.

This problem set is arranged so that odd- and even-numbered problems are roughly equivalent and so that problems progress from easy to hard. Apart from these criteria, there is no particular pattern to the arrangement. Students are expected to select the appropriate technique based on the merits of the problem.

Using a CAS to do the algebraic manipulation for problems in this section, students will be more confident as they approach the more difficult problems, developing critical thinking skills as they set up equations and systems of equations.

Students could use a system of equations to solve Problem 1. If x is the distance from the base of the mountain’s altitude to the 507 m segment, then tan 21.6° = \( \frac{h}{507 + x} \) and tan 35.8° = \( \frac{h}{x} \).

1. \( CD = 445.1 \text{ m} \)

2a. Window = 12.1 ft; Roof = 20.1 ft

2b. Area = 120.6 ft²

3a. =8.7 km

3b. \( A = 607.5 \text{ km}² \)

4a. =364 pumpkins

4b. \( \theta = 33.1229...° \)

5. =1380.6 yd or 608.4 yd

6. sin \( \theta = \frac{30 \sin 50°}{20} = 1.15 \), which is not the sine of any angle. It is impossible to build the truss to the specifications. The 20-ft side is too short, the 30-ft side is too large, or the 50° angle is too large.

7a. 0.6326... km \hspace{1cm} 7b. 0.1265... km/s

7c. 53.1210...°

8. =63.7 in. or 35.2 in.

See page 1020 for the answer to Problem Q6.
9a. \approx 463.4 \text{ km/h}
9b. \approx 537.0 \text{ km/h}

10a.

<table>
<thead>
<tr>
<th>\theta</th>
<th>L (lb)</th>
<th>H (lb)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0°</td>
<td>500000</td>
<td>0</td>
</tr>
<tr>
<td>5°</td>
<td>501910</td>
<td>43744</td>
</tr>
<tr>
<td>10°</td>
<td>507713</td>
<td>88163</td>
</tr>
<tr>
<td>15°</td>
<td>517638</td>
<td>133975</td>
</tr>
<tr>
<td>20°</td>
<td>532089</td>
<td>181985</td>
</tr>
<tr>
<td>25°</td>
<td>551689</td>
<td>233154</td>
</tr>
<tr>
<td>30°</td>
<td>577350</td>
<td>288675</td>
</tr>
</tbody>
</table>

10b. The centripetal force is stronger, so the plane is being forced more strongly away from a straight line into a circle.
10c. The horizontal component is 0, so there is no centripetal force to push the plane out of a straight path.
10d. \approx 33.56°
10e. The plane would start to fall and spiral downward.

11. Let \( F = \) the other person’s force. \( F = 66.0732\ldots \approx 66 \text{ lb} \). Then the magnitude of the resultant force vector is about 110.8 lb.

9. **Airplane Velocity Problem:** A plane is flying through the air at a speed of 500 km/h. At the same time, the air is moving at 40 km/h with respect to the ground at an angle of 23° with the plane’s path. The plane’s ground speed is the magnitude of the vector sum of the plane’s air velocity and the wind velocity. Find the plane’s ground speed if it is flying
   a. Against the wind
   b. With the wind

10. **Airplane Lift Problem:** When an airplane is in flight, the air pressure creates a force vector, called the lift, that is perpendicular to the wings. When the plane banks for a turn, this lift vector may be resolved into horizontal and vertical components. The vertical component has magnitude equal to the plane’s weight (this is what holds the plane up). The horizontal component is a centripetal force that makes the plane go on its curved path. Suppose that a jet plane weighing 500,000 lb banks at an angle \( \theta \) (Figure 9-7f).
   a. Make a table of magnitudes of lift and horizontal component for each 5° from 0° through 30°.
   b. Based on your table in part a, why can a plane turn in a smaller circle when it banks at a greater angle?
   c. Why does a plane fly straight when it is not banking?
   d. If the maximum lift the wings can sustain is 600,000 lb, what is the maximum angle at which the plane can bank?
   e. What might happen if the plane tried to bank at an angle greater than in part d?

11. **Canal Barge Problem:** In the past, it was common to pull a barge with tow ropes on opposite sides of a canal (Figure 9-7g). Assume that one person exerts a force of 50 lb at an angle of 20° with the direction of the canal. The other person pulls at an angle of 15° with respect to the canal with just enough force so that the resultant vector is directly along the canal. Find the force, in pounds, with which the second person must pull and the magnitude of the resultant force vector.

12. **Sailboat Force Vector Problem:** Figure 9-7h represents a sailboat with one sail, set at a 30° angle with the axis of the boat. The wind exerts a force vector of 300 lb that acts on the mast in a direction perpendicular to the sail.
   a. Find the absolute value of the component of the force vector along the axis of the boat. (This force makes the boat move forward.)
   b. How hard is the wind pushing the boat in the direction perpendicular to the axis of the boat? (The keel minimizes the effect of this force in pushing the boat sideways.)
   c. On the Internet or in some other reference source, look up the physics of sailboats to find out why two sails more than double the forward force produced by one sail. Give the source of your information.
12a. $|\text{axial component}| = 150 \text{ lb}$
12b. $|\text{normal component}| = 259.8076... \approx 260 \text{ lb}$
12c. Answers will vary, but the major effect is the increase in wind speed as the wind goes through the relatively narrow space between the sails, creating a thrust vector that has an additional component in the axial direction.

13a. $|\text{normal component}| = 38.974.8025... \approx 39,000 \text{ lb}$, which is not much less than the weight of the truck.
13b. $|\text{parallel component}| = 8998.0421... \approx 9000 \text{ lb}$, which is surprisingly large!
13c. Total force $= 98.8594... \approx 99 \text{ lb}$
13d. Answers will vary, but the major difference is that the coefficient of static friction is used to calculate the force necessary to start a stationary object moving, whereas the coefficient of dynamic friction, usually smaller, is used to calculate the force needed to keep an object in motion once it has been started.

14a. $y = 0.6x$
14b. $y = 53.4603... \approx 53.5 \text{ lb}$
14c. Total force $= 98.8594... \approx 99 \text{ lb}$
14d. Answers will vary, but the major difference is that the coefficient of static friction is used to calculate the force necessary to start a stationary object moving, whereas the coefficient of dynamic friction, usually smaller, is used to calculate the force needed to keep an object in motion once it has been started.
Problem Notes (continued)

15a. \( x = 5.7735... \approx 5.77 \text{ lb; } \) resultant force \( |\text{resultant force}| = 11.5470... \approx 11.55 \text{ lb} \)

15b. \( \theta = \tan^{-1} \frac{x}{10} \)

The graph shows that \( \theta \) approaches a horizontal asymptote at 90° as \( x \) gets larger.

15c. String tension = \( |\text{resultant force}| = \sqrt{10^2 + x^2} \)

The graph shows that the tension approaches \( x \) asymptotically as \( x \) gets larger.

16a. \( |\vec{t}_1| \cos 20^\circ = |\vec{t}_2| \cos 40^\circ \)

\( \Rightarrow |\vec{t}_1| \cos 20^\circ - |\vec{t}_2| \cos 40^\circ = 0 \)

16b. \( |\vec{t}_1| \sin 20^\circ + |\vec{t}_2| \sin 40^\circ = 50 \)

16c. \( |\vec{t}_1| = 44.2275... \approx 44.2 \text{ lb; } |\vec{t}_2| = 54.2531... \approx 54.3 \text{ lb} \)

16d. \( |\vec{t}_1| \cos 20^\circ = 44.2275... \cos 20^\circ = 41.5603... \text{ lb; } |\vec{t}_2| \cos 40^\circ = 54.2531... \cos 40^\circ = 41.5603... \text{ lb} \)

The two horizontal components are equal.

\( |\vec{t}_1| \sin 20^\circ + |\vec{t}_2| \sin 40^\circ = 44.2275... \sin 20^\circ + 54.2531... \sin 40^\circ = 50 \)

The two vertical components sum to 50.

16e. \( \vec{t}_2 \) bears more than twice the amount of the 50-lb weight as \( \vec{t}_1 \).

17. \( |\vec{t}_1| = 44.2275...; |\vec{t}_2| = 54.2531... \)

15. Hanging Weight Problem 1: Figure 9-7k shows a 10-lb weight hanging on a string 20 in. long. You pull the weight sideways with a force of magnitude \( x \), in pounds, making the string form an angle \( \theta \) with the vertical. In this problem you will find the measure of angle \( \theta \) as a function of how hard you pull and the resulting tension force in the string.

![Figure 9-7k](image)

a. The resultant force exerted on the string by the block is the vector sum of the 10-lb weight of the block and the \( x \)-lb force, and it acts in the direction of the string. With what force must you pull to make \( \theta = 30^\circ \)? What will be the tension in the string (the magnitude of the resultant vector)?

b. Write an equation expressing \( \theta \) as a function of \( x \). Sketch the graph of this function. What happens to the angle measure as \( x \) becomes very large?

c. Write another equation expressing the tension in the string as a function of \( x \). Sketch the graph of this function. What happens to this tension as \( x \) becomes very large?

16. Hanging Weight Problem 2: Figure 9-7l shows an object weighing 50 lb supported by two cables connected to walls 65 ft apart on opposite sides of an alley. Tension vectors \( \vec{t}_1 \) and \( \vec{t}_2 \) in the cables make angles of 20° and 40°, respectively, with the horizontal. The resultant vector of these tension vectors is the 50-lb vector pointed straight up, in a direction opposite to the weight vector. In this problem you will calculate the magnitudes of the two tension vectors.

![Figure 9-7l](image)

a. The horizontal components of vectors \( \vec{t}_1 \) and \( \vec{t}_2 \) have opposite directions but equal magnitudes. (Otherwise the object would move sideways!) Write an equation involving these magnitudes that expresses this fact.

b. The vertical components of \( \vec{t}_1 \) and \( \vec{t}_2 \) sum to the upward-pointing 50-lb vector. Write another equation involving the magnitudes of these tension vectors that expresses this fact.

c. Solve the system of equations in parts a and b to find the magnitudes of \( \vec{t}_1 \) and \( \vec{t}_2 \). Store the results without rounding.

d. Demonstrate numerically that the magnitudes of the horizontal components of \( \vec{t}_1 \) and \( \vec{t}_2 \) are equal and that the magnitudes of the vertical components sum to 50 lb.

e. Which tension vector bears more of the 50-lb weight, the one with the larger angle to the horizontal or the one with the smaller angle?

17. Hanging Weight by Law of Sines Problem: Figure 9-7m shows the two tension vectors \( \vec{t}_1 \) and \( \vec{t}_2 \) from Figure 9-7l drawn head-to-tail, with the 50-lb sum vector starting at the tail of \( \vec{t}_1 \) and ending at the head of \( \vec{t}_2 \). Use the law of sines to find the magnitudes of \( \vec{t}_1 \) and \( \vec{t}_2 \).

![Figure 9-7m](image)
18. **Ship’s Velocity Problem:** A ship is sailing through the water in the English Channel with velocity 22 knots on a bearing of 157°, as shown in Figure 9-7n. The current has velocity 5 knots on a bearing of 213°. The actual velocity of the ship is the vector sum of the ship’s velocity and the current’s velocity. Find the ship’s actual velocity.

![Figure 9-7n](image)

19. **Wind Velocity Problem:** A navigator on an airplane knows that the plane’s velocity through the air is 250 km/h on a bearing of 237°. By observing the motion of the plane’s shadow across the ground, she finds to her surprise that the plane’s ground speed is only 52 km/h and that its direction is along a bearing of 15°. She realizes that the ground velocity is the vector sum of the plane’s velocity and the wind velocity. What wind velocity would account for the observed ground velocity?

20. **Space Station Problem:** Ivan is in a space station orbiting Earth. He has the job of observing the motion of two communications satellites.
   a. As Ivan approaches the two satellites, he finds that one of them is 8 km away, the other is 11 km away, and the angle between the two (with Ivan at the vertex) is 120°. How far apart are the satellites?
   b. A few minutes later, Satellite 1 is 5 km from Ivan and Satellite 2 is 7 km from him. At this time, the two satellites are 10 km apart. At which of the three space vehicles does the largest angle of the resulting triangle occur? What is the measure of this angle? What is the area of the triangle?
   c. Several orbits later, only Satellite 1 is visible, while Satellite 2 is near the opposite side of Earth (Figure 9-7o). Ivan determines that the measure of angle A is 37.7°, the measure of angle B is 113°, and the distance between him and Satellite 1 is 4362 km. To the nearest kilometer, how far apart are Ivan and Satellite 2?

![Figure 9-7o](image)

21. **Visibility Problem:** Suppose that you are aboard a plane destined for Hawaii. The pilot announces that your altitude is 10 km. You decide to calculate how far away the horizon is. You draw a sketch as in Figure 9-7p and realize that you must calculate an arc length. You recall from geography that the radius of Earth is about 6400 km. How far away is the horizon along Earth’s curved surface? Is this surprising?

![Figure 9-7p](image)
Problem Notes (continued)
22a. \( x = 143.2665 \ldots \text{ cm or } 44.6719 \ldots \text{ cm} \)
22b. \((-200 \cos 50^\circ)^2 - 4 \cdot 1 \cdot 6400 = -9072.9635 < 0, \text{ so there is no possible solution. Or note that when } \theta = 50^\circ, \text{ the height of the hinge is } 100 \sin 50^\circ = 76.6044 \ldots \text{ cm, which is greater than the length of the second ruler.} \)
22c. \( 36.8698 \ldots \text{ m} \)
23a. \( = 133.4^\circ \)
23b. Area = 6838.2 m²
24a. \( = 4476.4 \text{ m}² \)
24b. \( = 137.5 \text{ m} \)
24c. \( = 43.0;\ = 58.0^\circ \)
25a. Answers will vary.
25b. The program should give the expected answer.
25c. Label the 95° angle \( A \), and label the rest of the vertices clockwise as \( B \) through \( F \): \( AC = \sqrt{20^2 + 22^2 - 2 \cdot 20 \cdot 22 \cos 114^\circ} = 35.2410 \ldots \text{ m} \)
\( \angle ACB = \sin^{-1} \left( \frac{20 \sin 114^\circ}{AC} \right) = 31.2287 \ldots^\circ \)
\( \angle ACD = 147^\circ - \angle ACB = 115.7712 \ldots^\circ \)
\( AD = \sqrt{AC^2 + 15^2 - 2 \cdot AC \cdot 15 \cos \angle ACD} = 43.8929 \ldots \text{ m} \)
\( \angle ADC = \sin^{-1} \left( \frac{15 \sin \angle ACD}{AD} \right) = 46.3050 \ldots^\circ \)
\( \angle ADE = 122^\circ - \angle ADC = 75.6949 \ldots^\circ \)
\( AE = \sqrt{AD^2 + 18^2 - 2 \cdot AD \cdot 18 \cos \angle ADE} = 43.1295 \ldots \text{ m} \)
\( \angle AED = \sin^{-1} \left( \frac{AD \sin \angle ADE}{AE} \right) = 80.4510 \ldots^\circ \)
\( \angle AEF = 115^\circ - \angle AED = 34.5489 \ldots^\circ \)
\( AF = \sqrt{AE^2 + 17^2 - 2 \cdot AE \cdot 17 \cos \angle AEF} = 30.6817 \ldots \text{ m} \)

22. **Hinged Rulers Problem:** Figure 9-7q shows a meterstick (100-cm ruler) with a 60-cm ruler attached to one end by a hinge. The other ends of both rulers rest on a horizontal surface. The hinge is pulled upward in such a way that the meterstick makes an angle \( \theta \) with the surface. \[ \text{Figure 9-7q} \]

(a) Find the two possible distances between the ruler ends if \( \theta = 20^\circ \).
(b) Show that there is no possible triangle if \( \theta = 50^\circ \).
(c) Find the value of \( \theta \) that gives just one possible distance between the ends.

23. **Surveying Problem 1:** A surveyor measures the three sides of a triangular field and gets lengths 114 m, 165 m, and 257 m.

(a) What is the measure of the largest angle of the triangle?
(b) What is the area of the field?

24. **Surveying Problem 2:** A field has the shape of a quadrilateral that is not a rectangle. Three sides measure 50 m, 60 m, and 70 m, and two angles measure 127° and 132° (Figure 9-7r).

(a) By dividing the quadrilateral into two triangles, find its area.
(b) Find the length of the fourth side.
(c) Find the measures of the other two angles.

25d. For a nonconvex polygon, you might not be able to divide it into triangles that fan out radially from a single vertex.

25. **Surveying Problem 3:** Surveyors find the area of an irregularly shaped tract of land by taking “field notes.” These notes consist of the length of each side and information for finding each angle measure. For this problem, starting at one vertex, the tract is divided into triangles. For the first triangle, two sides and the included angle are known (Figure 9-7s), so you can calculate its area. To calculate the area of the next triangle, you must recognize that one of its sides is also the third side of the first triangle and that one of its angles is an angle of the polygon (147° in Figure 9-7s) minus an angle of the first triangle. By calculating the measures of this side and angle and using the next side of the polygon (15 m in Figure 9-7s), you can calculate the area of the second triangle. The areas of the remaining triangles are calculated in the same manner. The area of the tract is the sum of the areas of the triangles.

(a) Write a program for calculating the area of a tract using the technique described. The input should be the measures of the sides and angles of the polygon. The output should be the area of the tract.
(b) Use your program to calculate the area of the tract in Figure 9-7s. If you get approximately 1029.69 m², you can assume that your program is working correctly.
(c) Show that the last side of the polygon has length 30.6817... m, which is close to the measured value, 31 m.
(d) The polygon in Figure 9-7s is a convex polygon because none of the angles measure more than 180°. Explain why your program might give wrong answers if the polygon were not convex.
Review Problems

R0. Update your journal with things you learned in this chapter. Include topics such as the laws of cosines and sines, the area formulas, how these are derived, and when it is appropriate to use them. Also include how triangle trigonometry is applied to vectors.

R1. Figure 9-8a shows triangles with sides 4 cm and 5 cm, with a varying included angle $\theta$. The length of the third side (dashed) is a function of $\theta$. The five values of $\theta$ shown are 30°, 60°, 90°, 120°, and 150°.

- a. Measure the length of the third side (dashed) for each triangle.
- b. How long would the third side be if the angle were 180°? If it were 0°?
- c. If $\theta = 90^\circ$, you can calculate the length of the dashed line by means of the Pythagorean theorem. Does your measured length in part a agree with this calculated length?
- d. If $y$ is the length of the dashed line, the law of cosines states that

$$y = \sqrt{5^2 + 4^2 - 2 \cdot 5 \cdot 4 \cdot \cos \theta}$$

Plot the data from parts a and b and this equation for $y$ on the same screen. Do the data seem to fit the law of cosines? Does the graph seem to be part of a sinusoid? Explain.

R2. a. Make a sketch of a triangle with sides 50 ft and 30 ft and included angle 153°. Find the length of the third side.
- b. Make a sketch of a triangle with sides 8 m, 5 m, and 11 m. Calculate the measure of the largest angle.
- c. Suppose you want to construct a triangle with sides 3 cm, 5 cm, and 10 cm. Explain why this is geometrically impossible. Show why computation of an angle using the law of cosines leads to the same conclusion.
- d. Sketch $\triangle DEF$ with angle $D$ in standard position in a uv-coordinate system. Find the coordinates of points $E$ and $F$ in terms of sides $e$ and $f$ and angle $D$. Use the distance formula to prove that you can calculate $d$ using

$$d^2 = e^2 + f^2 - 2ef \cos D$$

Section Notes

Section 9-8 contains a set of review problems, a set of concept problems, and a chapter test. The review problems include one problem for each section in the chapter. You may wish to use the chapter test as an additional set of review problems.

Encourage students to practice the no-calculator problems without a calculator so that they are prepared for the test problems for which they cannot use a calculator.

R1b. $5 + 4 = 9; 5 - 4 = 1$

R1c. $\sqrt{5^2 + 4^2} \approx 6.4; \text{ Yes}$

R1d.

No, the shape is not a sinusoid.
Differentiating Instruction

- Students should do the review problems in pairs. Go over the review problems in class, perhaps by having students present their solutions. You might assign students to write up their solutions before class starts.
- Work through the concept problems as a class activity to give students another opportunity to master the new vocabulary.
- For Problem C5, explain dot product and mention that there exists another type of vector product called the cross product.
- Model good explanations for Problems T4–T7, but take language difficulties into account when assessing student responses. Encourage students to use diagrams as part of their explanations.
- Even with language support, the cumulative review will probably take too much time for ELL students to complete. Allow students to work in pairs, and shorten the assignment.
- Because many cultures’ norms highly value helping peers, ELL students often help each other on tests. You can limit this tendency by making multiple versions of the test.
- Consider giving a group test the day before the individual test, so that students can learn from each other as they review, and they can identify what they don’t know prior to the individual test. Give a copy of the test to each group member, have them work together, then randomly choose one paper from the group to grade. Grade the test on the spot, so students know what they need to review further. Make this test worth \( \frac{1}{3} \) the value of the individual test, or less.
- ELL students may need more time to take the test.
- ELL students will benefit from having access to their bilingual dictionaries while taking the test.

R3. a. Make a sketch of a triangle with sides 50 ft and 30 ft and included angle 153°. Find the area of the triangle.
   b. Make a sketch of a triangle with sides 8 mi, 11 mi, and 15 mi. Find the measure of one angle and use it to find the area of the triangle. Calculate the area again using Heron’s formula. Show that the results are the same.
   c. Suppose that two sides of a triangle have lengths 10 yd and 12 yd and that the area is 40 yd². Find the two possible measures of the included angle between these two sides.
   d. Sketch \( \triangle DEF \) with side \( d \) horizontal. Draw the altitude from vertex \( D \) to side \( d \). What does this altitude equal in terms of side \( e \) and angle \( F \)? By appropriate geometry, show that the area of the triangle is
   \[
   \text{Area} = \frac{1}{2}de \sin F
   \]
R4. a. Make a sketch of a triangle with one side 6 in., the angle opposite that side 39°, and another angle, 48°. Calculate the length of the side opposite the 48° angle.
   b. Make a sketch of a triangle with one side 5 m and its two adjacent angles measuring 112° and 38°. Find the length of the longest side of the triangle.
   c. Make a sketch of a triangle with one side 7 cm, a second side 5 cm, and the angle opposite the 5-cm side 31°. Find the two possible measures of the angle opposite the 7-cm side.
   d. Sketch \( \triangle DEF \) and show sides \( d, e, \) and \( f \). Write the area three ways: in terms of angle \( D \), in terms of angle \( E \), and in terms of angle \( F \). Equate the areas and then perform calculations to derive the three-part equation expressing the law of sines.
R5. Figure 9-8b shows a triangle with sides 5 cm and 8 cm and angles \( \theta \) and \( \phi \), not included by these sides.

![Figure 9-8b](image-url)

\[ \theta \]
\[ \phi \]

If \( \theta = 22° \), calculate the two possible values of the length of the third side.
If \( \theta = 85° \), show algebraically that there is no possible triangle.
Calculate the value of \( \theta \) for which there is exactly one possible triangle.
If \( \phi = 47° \), calculate the one possible length of the third side of the triangle.

R6. a. Vectors \( \vec{a} \) and \( \vec{b} \) make a 174° angle when placed tail-to-tail (Figure 9-8c). The magnitudes of the vectors are \( | \vec{a} | = 6 \) and \( | \vec{b} | = 10 \). Find the magnitude of the resultant vector \( \vec{a} + \vec{b} \) and the angle this resultant vector makes with \( \vec{a} \) when they are placed tail-to-tail.

![Figure 9-8c](image-url)

b. Suppose that \( \vec{a} = 5\vec{i} + 3\vec{j} \) and \( \vec{b} = 7\vec{i} - 6\vec{j} \). Find the resultant vector \( \vec{a} + \vec{b} \) as sums of components. Then find the vector again as a magnitude and an angle in standard position.

c. A ship moves west (bearing of 270°) for 120 mi and then turns and moves on a bearing of 130° for another 200 mi. How far is the ship from its starting point? What is the ship’s bearing relative to its starting point?

d. A plane flies through the air at 300 km/h on a bearing of 220°. Meanwhile, the air is moving at 60 km/h on a bearing of 115°. Find the plane’s resultant ground velocity as a sum of two components, where unit vector \( \vec{i} \) points north and \( \vec{j} \) points east. Then find the plane’s resultant ground speed and the bearing on which it is actually moving.
e. **Calvin’s Roof Vector Problem**: Calvin does roof repairs. Figure 9-8d shows him sitting on a roof that makes an angle $\theta$ with the horizontal. The parallel component of his 160-lb weight vector acts to pull him down the roof. The frictional force vector counteracts the parallel component with magnitude $\mu$ (Greek letter $\mu$) times the magnitude of the normal component of the force vector. Here $\mu$ is the coefficient of friction, a nonnegative constant which is usually less than or equal to 1. If $\mu = 0.9$ and $\theta = 40^\circ$, will Calvin be able to sit on the roof without sliding? What is the steepest roof Calvin can sit on without sliding? Why couldn’t Calvin ever be held by friction alone on a roof with $\theta > 45^\circ$?

![Figure 9-8d](image)

**R7. Airport Problem (parts a–f):** Figure 9-8e shows Nagoya Airport and Tokyo Airport 260 km apart. The ground controllers at Tokyo Airport monitor planes within a 100-km radius of the airport.

a. Plane 1 is 220 km from Nagoya Airport at an angle of 32° to the straight line between the airports. How far is Plane 1 from Tokyo Airport? Is it really out of range of Tokyo Ground Control, as suggested by Figure 9-8e?

![Figure 9-8e](image)

b. Plane 2 is going to take off from Nagoya Airport and fly past Tokyo Airport. Its path will make an angle $\theta$ with the line between the airports. If $\theta = 15^\circ$, how far will Plane 2 be from Nagoya Airport when it first comes within range of Tokyo Ground Control? How far from Nagoya Airport is it when it is last within range? Store both of these distances in your calculator, without rounding.

c. Show that if $\theta = 40^\circ$, Plane 2 is never within range of Tokyo Ground Control.

d. Calculate the value of $\theta$ for which Plane 2 is within range of Tokyo Ground Control at just one point. How far from Nagoya Airport is this point? Store the distance in your calculator, without rounding.

e. Show numerically that the square of the distance in part d is exactly equal to the product of the two distances in part b. What theorem from geometry expresses this result?

f. Plane 3 (Figure 9-8e) reports that it is being forced to land on an island at sea! Nagoya Airport and Tokyo Airport report that the angle measures between Plane 3’s position and the line between the airports are 35° and 27°, respectively. Which airport is Plane 3 closer to? How much closer?

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**PROBLEM NOTES**

Supplementary problems for this section are available at [www.keypress.com/keyonline](http://www.keypress.com/keyonline).

R5b. $\sin \phi = 1.6$, which is not the sine of any angle.

R5c. The 5-cm side must be perpendicular to the third side, making the 8-cm side the hypotenuse of a right triangle. Then $\theta = \sin^{-1} \frac{5}{8} = 38.7^\circ$.

R5d. $= 10.5$ cm

R6a. $|\vec{r}| = 4.0813...; \phi = 165.2^\circ$

R6b. $\vec{a} + \vec{b} = 12\vec{r} - 3\vec{j}; |\vec{r}| = 12.4;$ $\theta = 346.0^\circ$

R6c. $|\vec{r}| = 132.7775... \text{ mi at a bearing of } 165.5160^\circ$

R6d. $\vec{r} = -138.4578...\vec{i} - 255.1704...\vec{j}; |\vec{r}| = 290.3145...\text{ at a bearing of } 208.4846^\circ$

R6e. Calvin will not slide down because the friction force is greater than the magnitude of the parallel component. The steepest angle is a bit less than 42°. If $\theta > 45^\circ$, then the parallel component has magnitude greater than that of the normal component, so friction alone could not keep Calvin from sliding down the roof.

R7a. $= 137.8$ km, so it is out of range.

R7b. 177.1700... km or 325.1113... km

R7c. $(-520 \cos 40^\circ)^2 - 4 \cdot 1 \cdot 57,600 = -71,722.7663...$, so $x$ is undefined.

R7d. The line from the plane to Tokyo Airport must be perpendicular to the flight path, so $x = 22.6^\circ$

R7e. $240^\circ = 57,600$  
$= (177.1700...)(325.1113...)$

The theorem states that if $P$ is a point exterior to circle $C$, $PR$ cuts $C$ at $Q$ and $R$, and $PS$ is tangent to $C$ at $S$, then $PQ \cdot PR = PS^2$.

R7f. Nagoya Airport is closer by about 35.2 km.

See page 1020 for the answers to Problem R4.

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Section 9-8: Chapter Review and Test 483
Problem Notes (continued)

R7g. \( = 7.6^\circ \sqrt{2} \)

R7h. \( \sqrt{3000^2 + 400^2} = 3026.5 \text{ lb} \)

R7i. The helicopter can tilt so that the thrust vector exactly cancels the wind vector.

C1. Student essay

C2a. \( 360^\circ - 250^\circ = 110^\circ \)
\( \sqrt{6^2 + 7^2 - 2 \cdot 6 \cdot 7 \cos 110^\circ} = 10.7 \text{ ft} \)
\( \sqrt{6^2 + 7^2 - 2 \cdot 6 \cdot 7 \cos 250^\circ} = 10.7 \text{ ft} \)
The answers are the same because \( \cos 250^\circ = \cos 110^\circ \).

C2b. \( \frac{1}{2} \cdot 6 \cdot 7 \sin 110^\circ = 19.7335... \text{ ft}^2 \)
\( \frac{1}{2} \cdot 6 \cdot 7 \sin 250^\circ = -19.7335... \text{ ft}^2 \)
The answers are opposite because \( \sin 250^\circ = -\sin 110^\circ \).

C2c. \( A_{\triangle ABC} = 50.3919... \text{ ft}^2; \)
\( A_{\triangle ABD} = 30.7 \text{ ft}^2 \)
Directly: First find \( \triangle C \).
\( DB^2 = 6^2 + 7^2 - 2 \cdot 6 \cdot 7 \cos 250^\circ \)
\( = 10^2 + 12^2 - 2 \cdot 10 \cdot 12 \cos C \)
\( \Rightarrow 240 \cos C = 159 + 84 \cos 250^\circ \)
\( \Rightarrow C = \cos^{-1} \left( \frac{159 + 84 \cos 250^\circ}{240} \right) \)
\( = 57.1260...^\circ \)
\( \Rightarrow \frac{1}{2} \cdot 10 \cdot 12 \sin C + \frac{1}{2} \cdot 6 \cdot 7 \sin 250^\circ = 30.7 \text{ ft}^2 \)

C3. Student project

Helicopter Problem (parts g-i): The rotor on a helicopter creates an upward force vector (Figure 9-8f). The vertical component of this force (the lift) balances the weight of the helicopter and keeps it in the air. The horizontal component (the thrust) makes the helicopter move forward. Suppose that the helicopter weighs 3000 lb.

**g.** At what angle will the helicopter have to tilt forward to create a thrust of 400 lb?

**h.** What will be the magnitude of the total force vector?

**Concept Problems**


C2. Reflex Angle Problem: Figure 9-8g shows quadrilateral \( ABCD \), in which angle \( A \) is a reflex angle measuring 250°. The resulting figure is called a nonconvex polygon. Note that the diagonal from vertex \( B \) to \( D \) lies outside the figure.

C3. Angle of Elevation Experiment: Construct an inclinometer that you can use to measure angles of elevation. One way to do this is to hang a piece of wire, such as a straightened paper clip, from the hole in a protractor, as shown in Figure 9-8h. Then tape a straw to the protractor so that you can sight a distant object more accurately. As you view the top of a building or tree along the straight edge of the protractor, gravity holds the paper clip vertical, allowing you to determine the angle of elevation.
Use your apparatus to measure the height of a tree or building using the techniques of this chapter.

C4. Euclid’s Problem: This problem comes from Euclid’s Elements. Figure 9-8i shows a circle with a secant line and a tangent line.

a. Sketch a similar figure using a dynamic geometry program, such as The Geometer’s Sketchpad, and measure the lengths of the secant segments, PQ and PR, and the tangent segment PS. By varying the radius of the circle and the angle QPO, see if it is true that $PS^2 = PQ \cdot PR$

b. Using the trigonometric laws and identities you’ve learned, prove that the equation in part a is a true statement.

C5. Dot (Scalar) Product of Two Vectors Problem: Figure 9-8j shows two vectors in standard position:
\[
\vec{a} = 3\hat{i} + 4\hat{j} \\
\vec{b} = 7\hat{i} + 2\hat{j}
\]

The dot product, written $\vec{a} \cdot \vec{b}$, is defined to be
\[
\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \theta
\]

where $\theta$ is the angle between the two vectors when they are placed tail-to-tail. Find the measure of the angle between $\vec{a}$ and $\vec{b}$, and store it without rounding. Use the result and the exact lengths of $\vec{a}$ and $\vec{b}$ to calculate $\vec{a} \cdot \vec{b}$. You should find that the answer is an integer! Figure out a way to calculate $\vec{a} \cdot \vec{b}$ using only the coefficients of the unit vectors: 3, 4, 7, and 2. Why do you suppose the dot product is also called the scalar product of the two vectors?

\[
\theta = 37.1847... ^\circ; \vec{a} \cdot \vec{b} = 29
\]

The dot product can also be calculated by finding the sum of the products of the $i$ coefficients and the $j$ coefficients: $\vec{a} \cdot \vec{b} = 3 \cdot 7 + 4 \cdot 2 = 29$. This method is covered in Chapter 12. The dot product is called the scalar product because the answer is a scalar, not a vector.
Problem Notes (continued)

T1. \[ d^2 = c^2 + e^2 - 2ce \cos D \]
T2. \[
\frac{c}{\sin C} = \frac{d}{\sin D} = \frac{e}{\sin E}
\]
or
\[
\frac{c}{\sin C} = \frac{d}{\sin D} = \frac{e}{\sin E}
\]

T3. \[ A = \frac{1}{2}de \sin C \]
T4. ASA is shown, but the law of cosines works only for ASA, SAA, and SSA.
T5. SAS is shown, but the law of sines works only for ASA, SAA, and SSA.
T6. \[ 10 + 7 < 19 \]
T7. The range of \( \cos^{-1} \) is \( 0^\circ \leq \theta \leq 180^\circ \), which includes every possible angle measure for a triangle. But the range of \( \sin^{-1} \) is \( -90^\circ \leq \theta \leq 90^\circ \), so the function \( \sin^{-1} \) cannot find obtuse angles.
T8.

\[ \begin{array}{c}
\pi \\
\pi + \theta
\end{array} \]

T9.

\[ \begin{array}{c}
\theta \\
\theta + \theta
\end{array} \]

T10. Student drawing. The third side should be about 3.2 cm.
T11. =3.2 cm
T12.

\[ \begin{array}{c}
38^\circ \\
50
\end{array} \]

The third angle measures 95°.
T17. Figure 9-8p shows a circle of radius 3 cm. Point P is 5 cm from the center. From point P, a secant line is drawn at an angle of 26° to the line connecting the center to P. Use the law of cosines to calculate the two unknown lengths labeled a and b in the figure.

![Figure 9-8p](image)

T18. Recall that the radius of a circle drawn to the point of tangency is perpendicular to the tangent. Use this fact to calculate the length of the tangent segment from point P in Figure 9-8p.

T19. Show numerically that the product of the two lengths you found in Problem T17 equals the square of the tangent length you found in Problem T18. This geometrical property appears in Euclid's Elements.

T20. For \( \vec{v} = 3\hat{i} - 5\hat{j} \), calculate the magnitude. Calculate the direction as an angle in standard position.

T21. Vector Difference Problem: Figure 9-8q shows position vectors

\[
\vec{a} = 3\hat{i} + 4\hat{j} \\
\vec{b} = 7\hat{i} + 2\hat{j}
\]

By subtracting components, find the difference vector, \( \vec{d} = \vec{a} - \vec{b} \). On a copy of Figure 9-8q, show that \( \vec{d} \) is equal to the displacement vector from the head of \( \vec{b} \) to the head of \( \vec{a} \). Explain how this interpretation of a vector difference is analogous to the way you determine how far your car has gone by subtracting the beginning odometer reading from the ending odometer reading.

![Figure 9-8q](image)

T22. What did you learn as a result of taking this test that you did not know before?

T13. \( \approx 30.9 \text{ ft} \)

T14–T16. Answers will vary.

T17. 6.5423... cm or 2.4456... cm

T18. 4 cm

T19. \((6.5423...)(2.4456...) = 16 = 4^2\)

T20. \( r = 5.8; \theta = 301.0^\circ \)

A blackline master for Problem T21 is available in the Instructor’s Resource Book.

T21. \(-4\hat{i} + 2\hat{j}\)

The graph shows that \( \vec{d} \) equals the displacement from the head of \( \vec{b} \) to the head of \( \vec{a} \), analogous to “where you end minus where you began.”

![Figure 9-8q](image)

T22. Answers will vary.
**Section 9-9**

**PLANNING**

**Class Time**
1 or 2 days

**Homework Assignment**
Day 1: Keep working on the Cumulative Review, Problems 19–37
Day 2: Complete the Cumulative Review, Problems 38–46, and do Problem Set 10-1

**Teaching Resources**
Blackline Master
Problem 45d
Test 26, Cumulative Test, Chapters 5–9, Forms A and B

**TEACHING**

**Section Notes**

The cumulative review questions in this section will help students rehearse for an exam on the trigonometric functions unit. Whenever possible, the problems are applied to real-world situations.

A cumulative exam can be quite an ordeal for students. Students working in small groups can have fun with these cumulative review problems and learn a lot from each other. Students should also be encouraged to look over their old tests and quizzes and bring in any problems they still don’t understand. You may also want to make up an additional set of practice problems that complement this problem set. Students should consult their journals to aid in the review process.

If you are giving a cumulative test, use this problem set as a guide for the type of problems to include. Use your judgment about the kind of review you will provide and the kind of cumulative exam you will give your students.

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**9-9 Cumulative Review, Chapters 5–9**

These problems constitute a 2- to 3-hour “rehearsal” for your examination on the trigonometric functions unit, Chapters 5–9. You began by studying periodic functions.

**Review Problems**

1. **Satellite Problem 1:** A satellite is in a circular orbit around Earth. From where you are on Earth’s surface, the straight-line distance to the satellite (through Earth, at times) is a periodic function of time. Sketch a reasonable graph.

To write equations for periodic functions such as the one in Problem 1, you generalized the trigonometric functions from geometry by allowing angles to be negative or greater than 180°.

2. Sketch a −213° angle in standard position. Draw the reference triangle and find the measure of the reference angle.

3. The terminal side of angle θ contains the point (12, −5) in the uv-coordinate system. Write the exact values (no decimals) of the six trigonometric functions of θ.

4. Write the exact value (no decimals) of sin 240°.


If θ is allowed to take on any real number of degrees, the trigonometric functions become periodic functions of θ.

6. Sketch the graph of the parent sine function, \( y = \sin \theta \).

7. What special name is given to the kind of periodic function you graphed in Problem 6?

Periodic functions such as the one in Problem 1 have independent variables that can be time or distance, not an angle measure. So you learned about circular functions whose independent variable is \( x \), not \( \theta \). The radian is the link between trigonometric functions and circular functions.

8. How many radians are in 360°? 180°? 90°? 45°?

9. How many degrees are in 2 radians?

10. Sketch a graph showing the unit circle centered at the origin of a uv-coordinate system. Sketch an x-axis tangent to the circle, going vertically through the point \( (u, v) = (1, 0) \). If the x-axis is wrapped around the unit circle, show that the point \( (2, 0) \) on the x-axis corresponds to angle measure 2 radians.

11. Sketch the graph of the parent circular sinusoidal function \( y = \cos x \).

Translation and dilation transformations also apply to circular function sinusoids.

12. For \( y = 3 + 4 \cos(5x + 6) \), find
   - a. The horizontal dilation
   - b. The vertical dilation
   - c. The horizontal translation
   - d. The vertical translation

13. For sinusoids, list the special names given to
   - a. The horizontal dilation
   - b. The vertical dilation
   - c. The horizontal translation
   - d. The vertical translation
To use sinusoids as mathematical models, you learned to write a particular equation from the graph.

![Graph](Figure 9-9a)

14. Write the particular equation for the sinusoid in Figure 9-9a.

15. If the graph in Problem 14 were plotted on a wide-enough domain, predict y for x = 342.7.

16. For the sinusoid in Problem 14, find algebraically the first three positive values of x if y = 4.

17. Show graphically that the three values you found in Problem 16 are correct.

18. Satellite Problem 2: Assume that in Problem 1, the satellite's distance varies sinusoidally with time. Suppose that the satellite is closest, 1000 mi from you, at time t = 0 min. Half a period later, at t = 50 min, it is at its maximum distance from you, 9000 mi. Write a particular equation for distance, in thousands of miles, as a function of time.

Radian gave you a convenient way to analyze the motion of two or more rotating objects.

19. Figure 9-9b shows a 5-cm-radius gear on a machine tool driving a 12-cm-radius gear. The design engineers want the smaller gear's teeth to have linear velocity 120 cm/s.

![Gear](Figure 9-9b)

Next you learned some properties of trigonometric and circular functions.

20. There are three kinds of properties that involve just one argument. Write the name of each kind of property, and give an example of each.

21. Use the properties in Problem 21 to prove that this equation is an identity. What restrictions are there on the domain of x?

\[
\sec^2 x \sin^2 x + \tan^4 x = \frac{\sin^2 x}{\cos^4 x}
\]

22. Reciprocal properties:

- \( \sec \theta = \frac{1}{\cos \theta}, \csc \theta = \frac{1}{\sin \theta} \)
- \( \cot \theta = \frac{1}{\tan \theta} \)

Quotient properties:

- \( \tan \theta = \frac{\sin \theta}{\cos \theta}, \cot \theta = \frac{\cos \theta}{\sin \theta} \)

Pythagorean properties:

- \( \sin^2 \theta + \cos^2 \theta = 1, \tan^2 \theta + 1 = \sec^2 \theta, 1 + \cot^2 \theta = \csc^2 \theta \)

- \( \sec^2 x \sin^2 x + \tan^4 x \)

For \( \cos x \neq 0 \left( x \neq \frac{\pi}{2} + \pi n \right) \)

See page 1021 for answers to Problems 5 and 10.
Problem Notes (continued)

23. Other properties involve functions of a composite argument. Write the composite argument property for \( \cos(x - y) \). Then express this property verbally.

24. Show numerically that \( \cos 34^\circ = \sin 56^\circ \).

25. Use the property in Problem 23 to prove that the equation \( \cos(90^\circ - \theta) = \sin \theta \) is an identity. How does this explain the result in Problem 24?

The properties can be used to explain why certain combinations of graphs come out the way they do.

26. Show that the function

\[
y = 3 \cos \theta + 4 \sin \theta
\]

is a sinusoid by finding algebraically the amplitude and phase displacement with respect to \( y = \cos \theta \) and writing \( y \) as a single sinusoid.

27. The function

\[
y = 12 \sin \theta \cos \theta
\]

is equivalent to the sinusoid \( y = 6 \sin 2\theta \). Prove algebraically that this is true by applying the composite argument property to \( \sin 2\theta \).

28. Write the double argument property expressing \( \cos 2x \) in terms of \( \sin x \) alone. Use this property to show algebraically that the graph of \( y = \sin^2 x \) is a sinusoid.

Sums and products of sinusoids with different periods have interesting wave patterns. By using harmonic analysis, you can write equations of the two sinusoids that were added or multiplied.

29. Find the particular equation for the function in Figure 9-9d.

30. Find the particular equation for the function in Figure 9-9e.

A product of sinusoids with very different periods can be transformed to a sum of sinusoids with nearly equal periods.

31. Transform the function

\[
y = 2 \cos 2\theta \cos \theta
\]

into a sum of two cosine functions.

32. Find the periods of the two sinusoids in the equation given in Problem 31 and the periods of the two sinusoids in the answer. What can you tell about relative sizes of the periods of the two sinusoids in the given equation and about relative sizes of the periods of the sinusoids in the answer?

Trigonometric and circular functions are periodic, so there are many values of \( \theta \) or \( x \) that give the same value of \( y \). Thus, the inverses of these functions are not functions.

33. Find the (one) value of the inverse trigonometric function \( \theta = \tan^{-1} 5 \).

34. Find the general solution of the inverse trigonometric relation \( x = \arcsin 0.4 \).

35. See Figure 9-9f in the student text. Possible parametric equations:

\[
x = \cos t, \quad y = t
\]

36. Domain is \(-1 \leq x \leq 1\);

Range is \(0 \leq y \leq \pi\)

37. \( \theta = 63.4^\circ, 243.4^\circ, 423.4^\circ, 603.4^\circ \)
Parametric functions make it possible to plot the graphs of inverse circular relations.

35. Use parametric functions to create the graph of \( y = \arccos x \), as shown in Figure 9-9f.

36. The inverse trigonometric function \( y = \cos^{-1} x \) is the principal branch of \( y = \arccos x \). Define the domain and range of \( y = \cos^{-1} x \).

37. Find the first four positive values of \( \theta \) if \( \theta = \arctan 2 \).

Last, you studied triangle and vector problems.

38. State the law of cosines.

39. State the law of sines.

40. State the area formula for a triangle given two sides and the included angle.

41. If a triangle has sides 6 ft, 7 ft, and 12 ft, find the measure of the largest angle.

42. Find the area of the triangle in Problem 41 using Heron’s formula.

43. Given \( \vec{a} = -3\mathbf{i} + 4\mathbf{j} \) and \( \vec{b} = 5\mathbf{i} + 12\mathbf{j} \),
   a. Find the resultant vector, \( \vec{a} + \vec{b} \), in terms of its components.
   b. Find the magnitude and angle in standard position of the resultant vector.
   c. Sketch a figure to show \( \vec{a} + \vec{b} \) added geometrically, head-to-tail.
   d. Is this true or false?
      \[ |\vec{a} + \vec{b}| = |\vec{a}| + |\vec{b}| \]
      Explain why your answer is reasonable.

44. Satellite Problem 4: In Problem 18, you assumed that the distance between you and the satellite was a sinusoidal function of time. In this problem you will get a more accurate mathematical model.

   a. Use the law of cosines and the distances in Figure 9-9g to find \( y \) as a function of angle \( x \), in radians.

   b. Use the fact that it takes 100 min for the satellite to make one orbit to write the equation for \( y \) as a function of time \( t \). Assume that \( x = 0 \) at time \( t = 0 \) min.

   c. Plot the equation from part b and the equation from Problem 18 on the same screen, thus showing that the functions have the same high points, low points, and period but that the equation from part b is not a sinusoid.

38. In \( \triangle ABC \), \( c^2 = a^2 + b^2 - 2ab \cos C \) (and similarly for \( a^2 \) and \( b^2 \)). The square of one side of a triangle is the sum of the squares of the other two sides minus twice their product times the cosine of the angle between them.

39. In \( \triangle ABC \), \( \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \).
    The length of one side of a triangle is to the sine of the angle opposite it as the length of any other side is to the sine of the angle opposite that side.

40. \( A_{\triangle ABC} = \frac{1}{2}ab \sin C = \frac{1}{2}bc \sin A = \frac{1}{2}ca \sin B \).
    The area of a triangle is \( \frac{1}{2} \) the product of any two sides and the sine of the angle between them.

41. \( \approx 134.6^\circ \)

42. 14,978... ft²

43a. \( 2\mathbf{i}^2 + 16\mathbf{j}^2 \)

43b. \( |\vec{r}| = 16.1; \theta \approx 82.9^\circ \)

43c. False. This is true only if \( \vec{a}^2 \) and \( \vec{b}^2 \) are at the same angle.

44. In units of 1000 miles:

44a. \( y = \sqrt{41 - 40 \cos x} \)

44b. \( y = \sqrt{41 - 40 \cos \frac{x^2}{50}} \).

44c. The dashed curve represents the equation from Problem 18.
45. **Three Force Vectors Problem:** Figure 9-9h shows three force vectors, \(\vec{a}, \vec{b}, \text{ and } \vec{c}\), acting on a point at the origin.

\[\vec{a} = 4\vec{i} + 3\vec{j}; \quad \vec{b} = -9\vec{i} + 4\vec{j}; \quad \vec{c} = 2\vec{i} - 5\vec{j}\]

\[\vec{d} = -3\vec{i} + 2\vec{j}\]

\[|\vec{d}| = \sqrt{13} \approx 3.6 \text{ newtons; } \theta \approx 146.3^\circ\]

a. Write each of the three vectors in terms of the unit vectors \(\vec{i}\) and \(\vec{j}\).

b. Find the resultant vector, \(\vec{d} = \vec{a} + \vec{b} + \vec{c}\), in terms of the unit vectors \(\vec{i}\) and \(\vec{j}\).

c. If the forces are measured in newtons, write the resultant force vector as a magnitude and a direction angle.

d. On a copy of Figure 9-9h, draw the three vectors head-to-tail in the order \((\vec{a} + \vec{b}) + \vec{c}\). Show that the resultant vector agrees with your answer to part b. Measure the magnitude and angle of the resultant vector with a ruler and protractor. Show that the results agree with part c.

It is important for you to be able to state verbally the things you have learned.

46. What do you consider to be the one most important thing you have learned so far as a result of studying precalculus?